Duality and Correspondence

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Dualities, informally:

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Back-and-forth mappings between classes of mathematical objects that induce systematic translations of their properties

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simplest examples,
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mechanic generation
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Linear algebra:
central to theory
of vector spaces

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Back-and-forth mappings between classes of mathematical objects that induce systematic translations of their properties

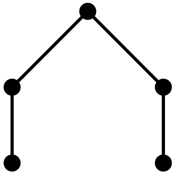
Order theory:
simplest examples,
self-dual objects

Projective geometry:
mechanic generation
of theorems

Linear algebra:
central to theory
of vector spaces

Logic:
generate new s&c theorems
Correspondence theory

Partial orders



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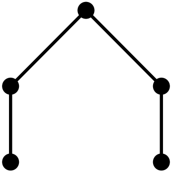

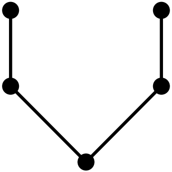

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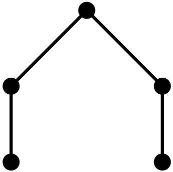

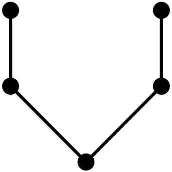

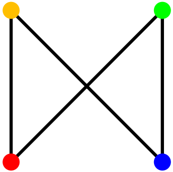

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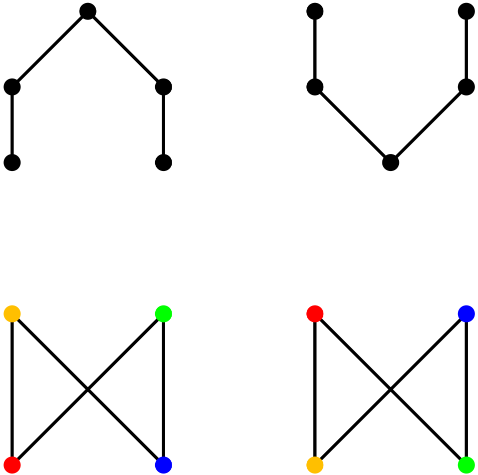

Through the looking glass

Dualities preserve information while reversing perspective.

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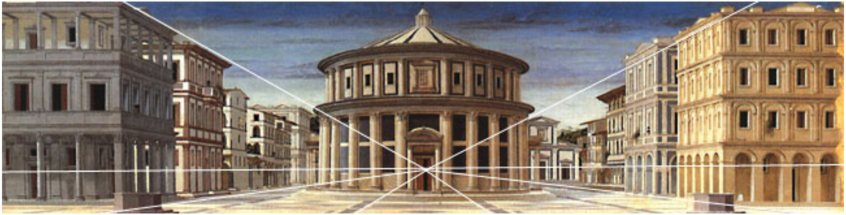
Dualities preserve information while reversing perspective.

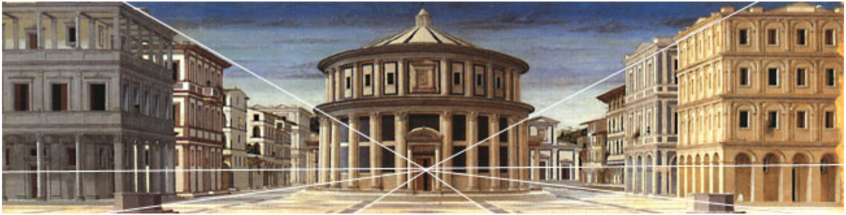
Self duality is some form of symmetry.

Projective geometry



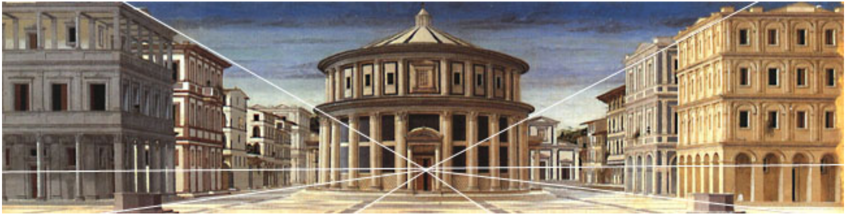
How to represent a three-dimensional space onto a plane?

Projective geometry



How to represent a three-dimensional space onto a plane?
Insight from perspective art: parallel lines meet at infinity.

Projective geometry



How to represent a three-dimensional space onto a plane?

Insight from perspective art: parallel lines meet at infinity.

Projective geometry formalizes this insight.

Axiomatizing the Projective Plane

Axiom 1

Axiom 2

Axiom 3

Axiom 4

Axiom 5

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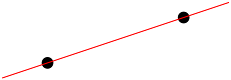

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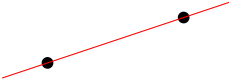

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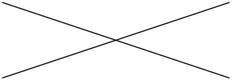

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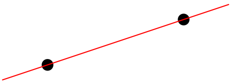

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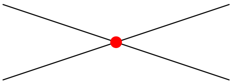

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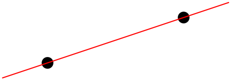

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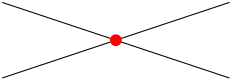

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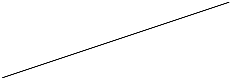

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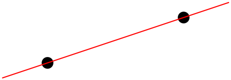

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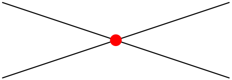

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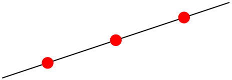

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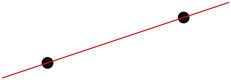

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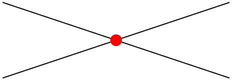

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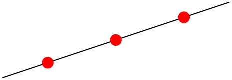

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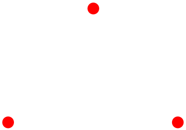

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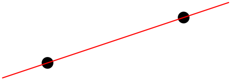

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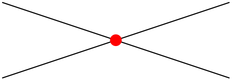

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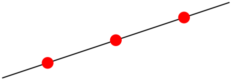

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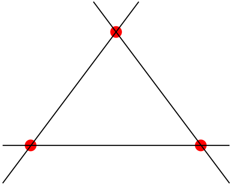

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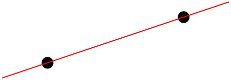

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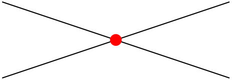

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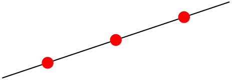

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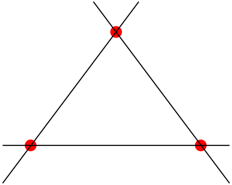

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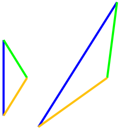

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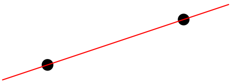

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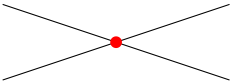

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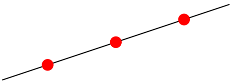

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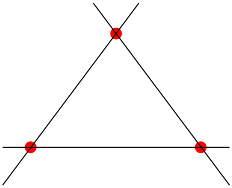

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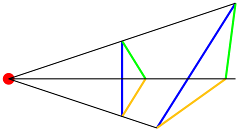

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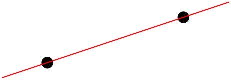

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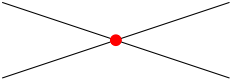

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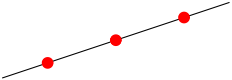

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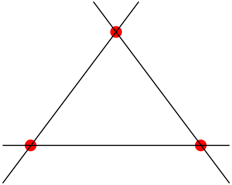

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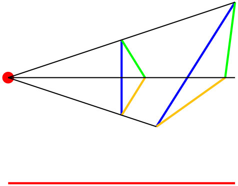

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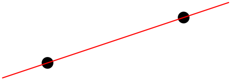

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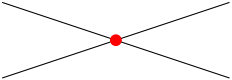

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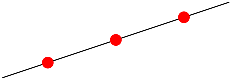

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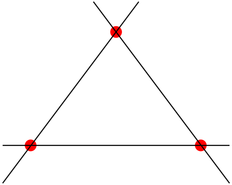

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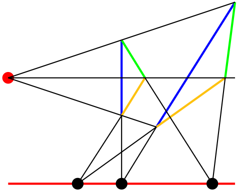

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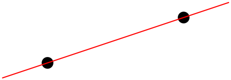

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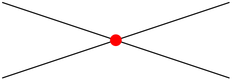

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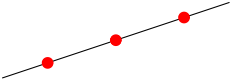

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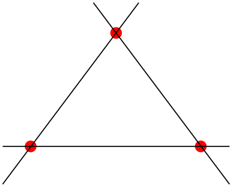

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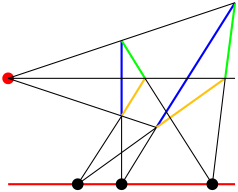

Minimal model

Axiomatizing the Projective Plane

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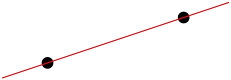

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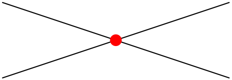

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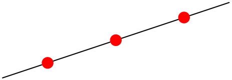

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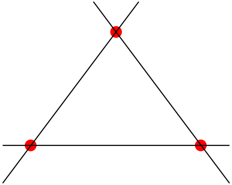

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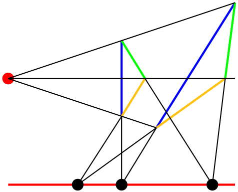

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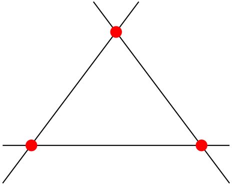

Axiomatizing the Projective Plane

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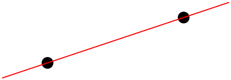

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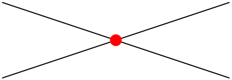

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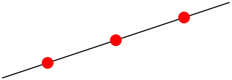

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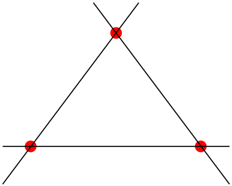

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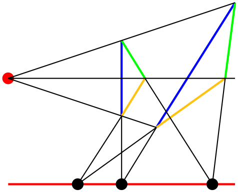

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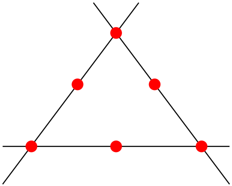

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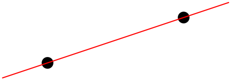
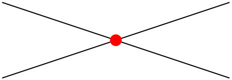


Axiom 2



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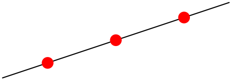

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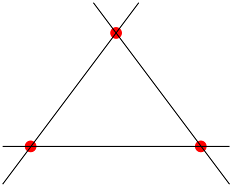

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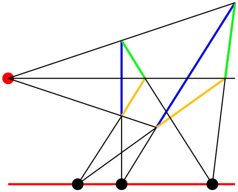
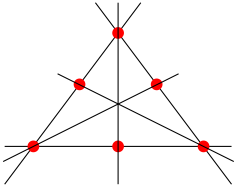


Minimal model



Axiomatizing the Projective Plane

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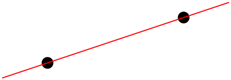

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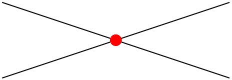

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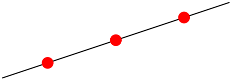

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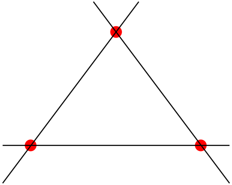

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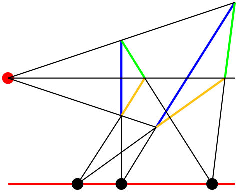

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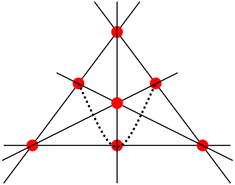

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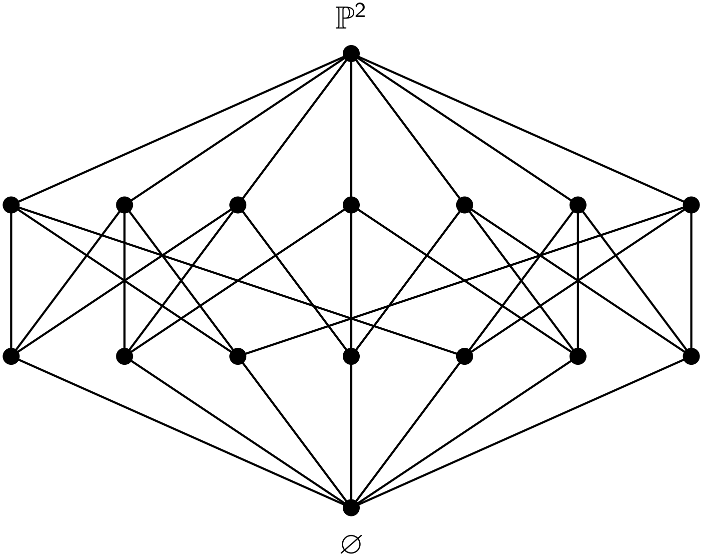

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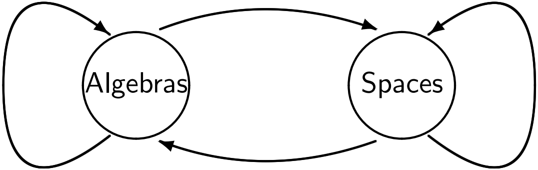

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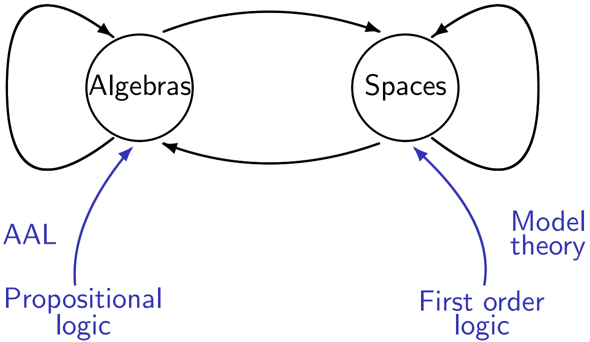

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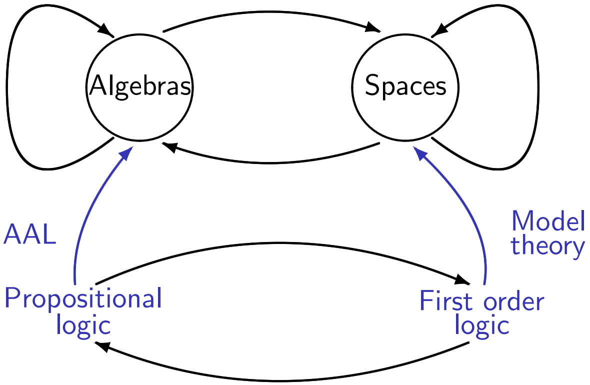

Dualities in Logic: a Rosetta stone

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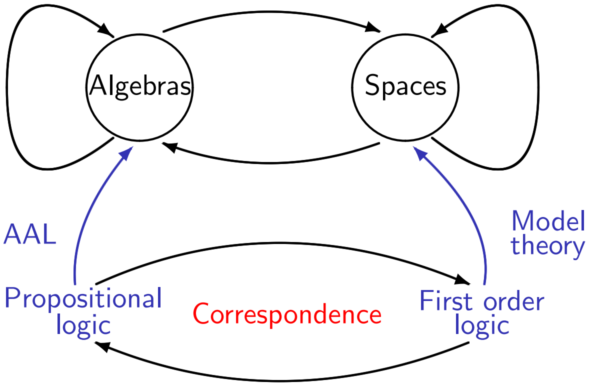

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"A smaller than B" $\neq$ "B larger than A"

$(\vee, \top)$-Subalgebra but not quotient

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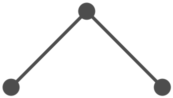

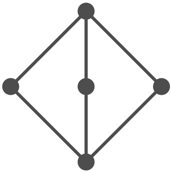

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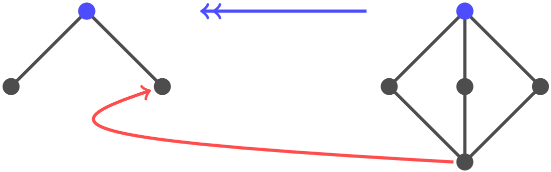

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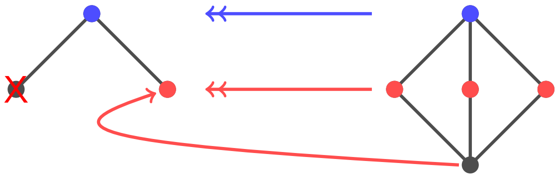

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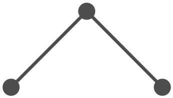

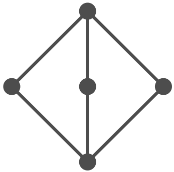

$(\vee, \top)$-Quotient but not subalgebra

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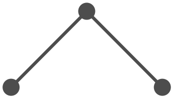

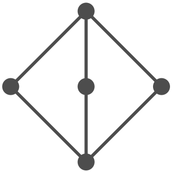

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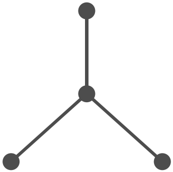

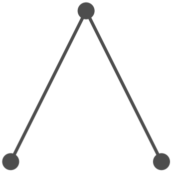

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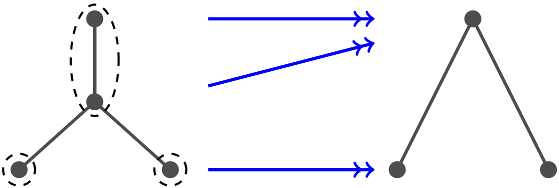

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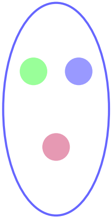

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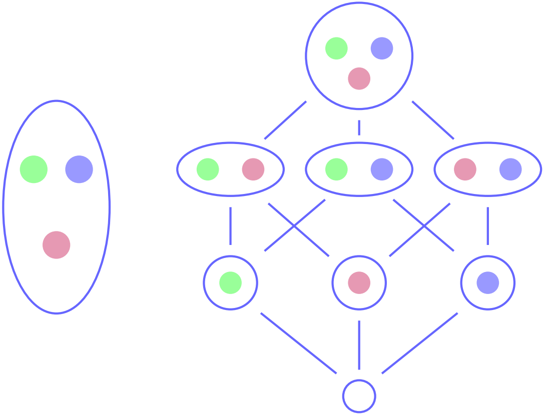

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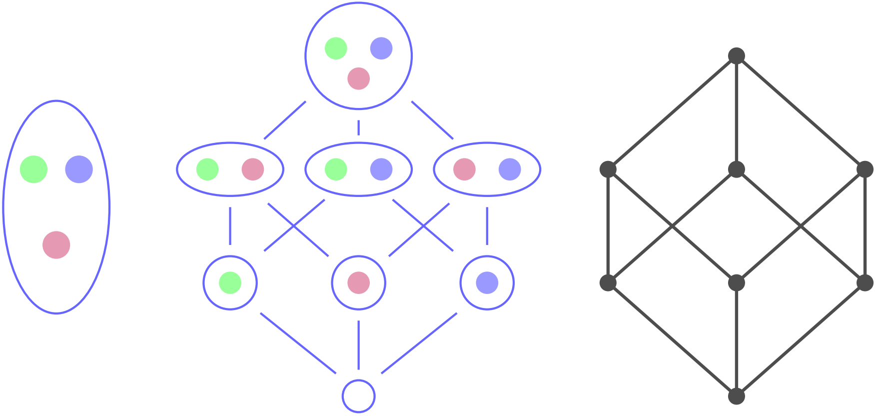

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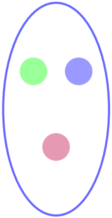

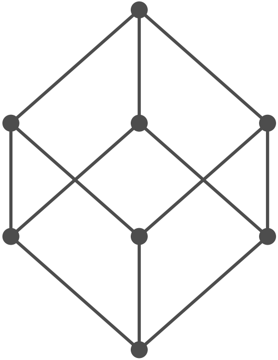

Duality extended to Maps

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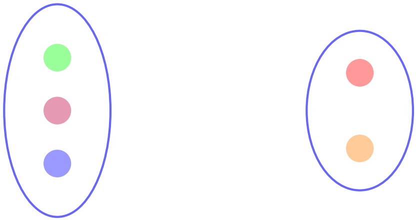

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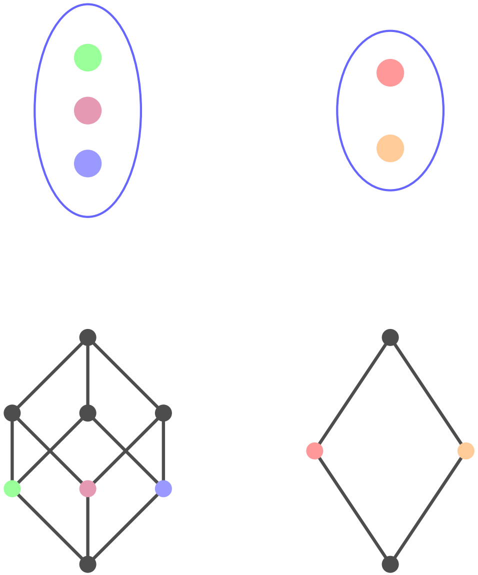

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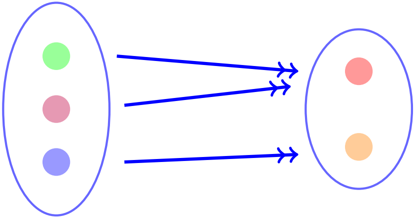

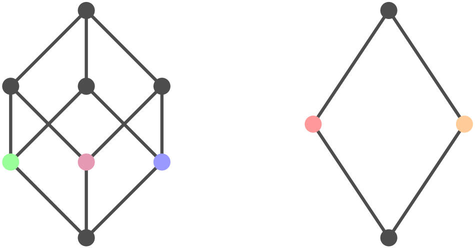

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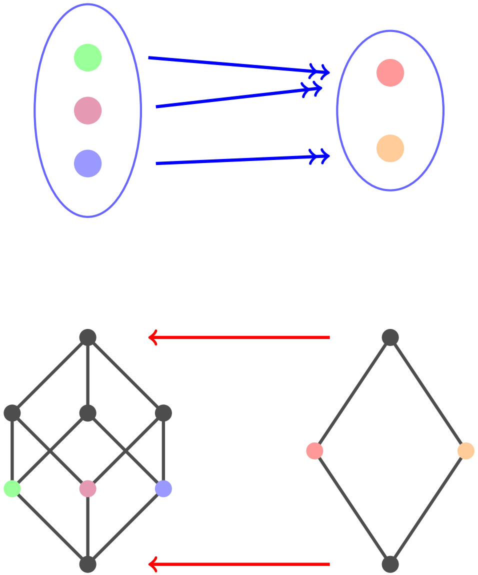

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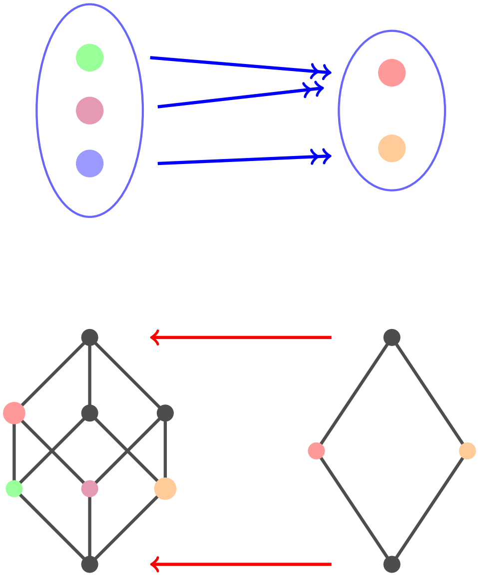

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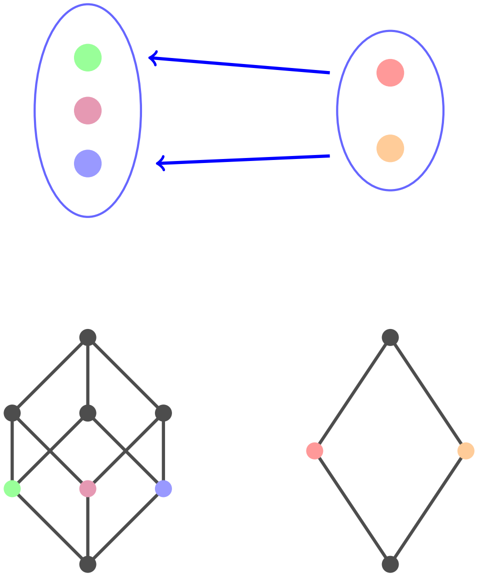

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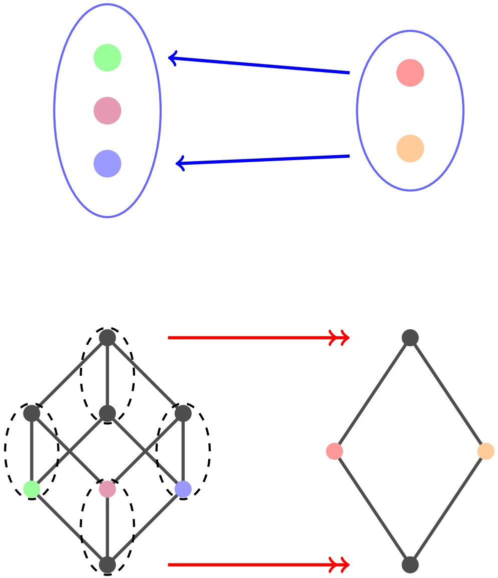

Expanded duality: from Kripke frames to BAOs

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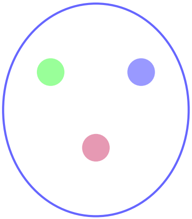

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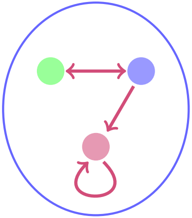

$\mathcal{R}$

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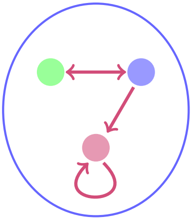

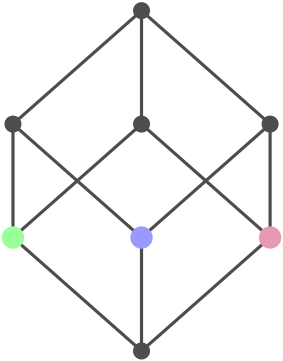

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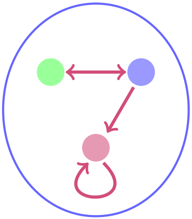

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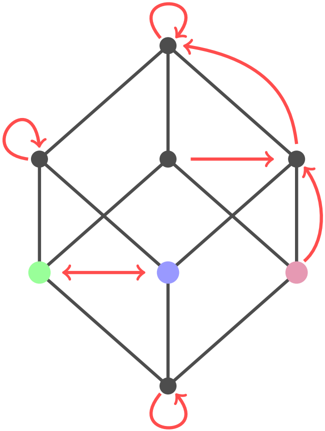

$\Diamond X = \mathcal{R}^{-1}[X]$

Expanded duality: from BAOs to Kripke frames

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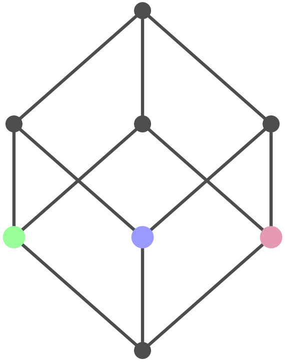

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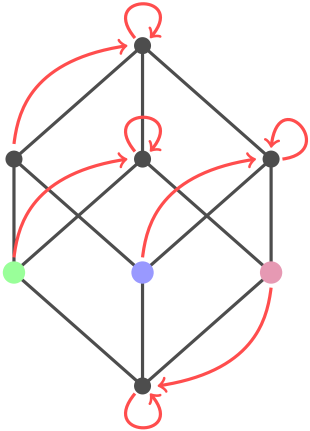

$$\Diamond : A \longrightarrow A, \ \Diamond\bot = \bot$$
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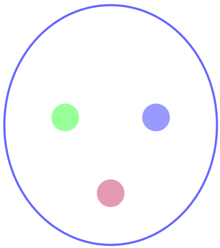

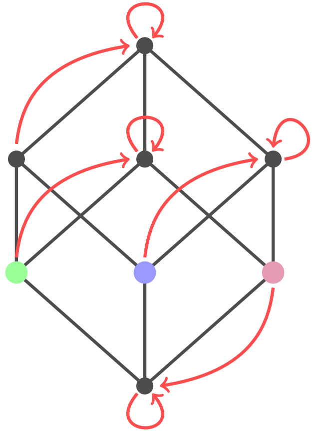

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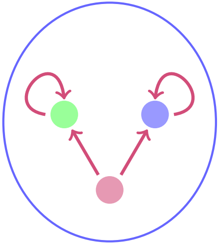
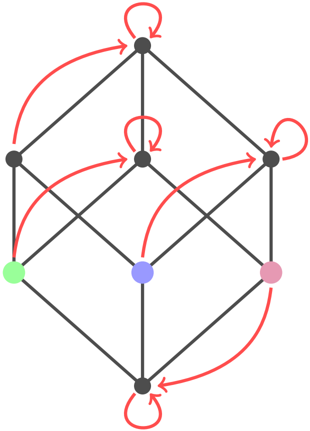


$$x\mathcal{R}y$$
$$x \leq \Diamond y$$



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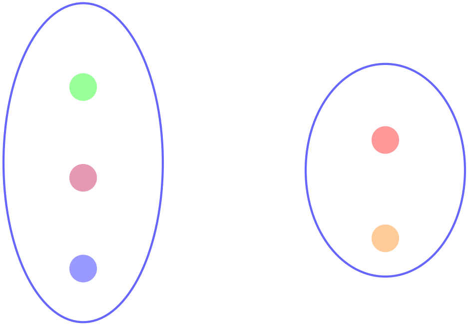

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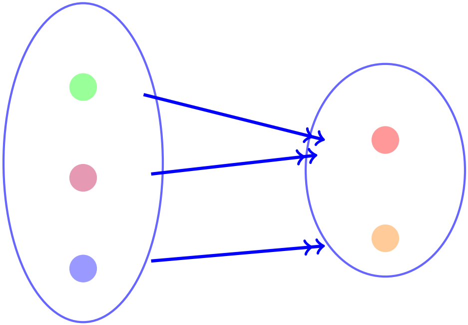

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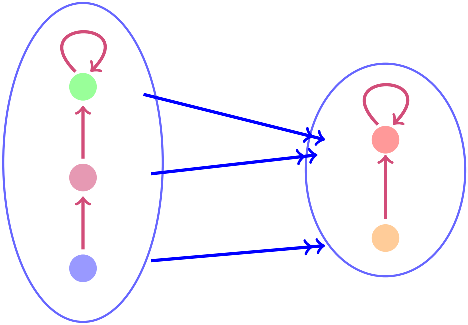

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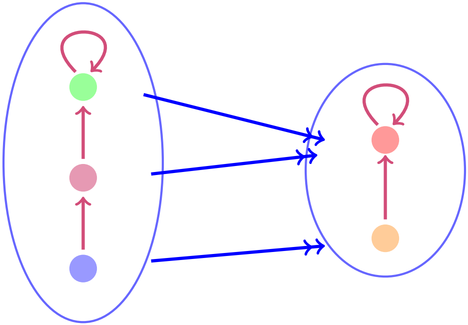
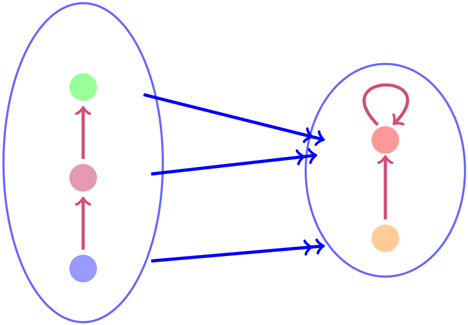


forth: $x\mathcal{R}y \Rightarrow f(x)\mathcal{R}'f(y)$

back: $f(x)\mathcal{R}'w \Rightarrow x\mathcal{R}y$ for some $y : f(y) = w$

Bounded morphisms



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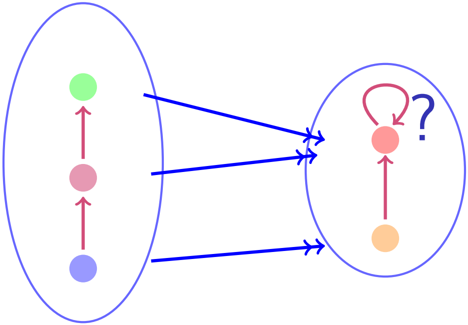

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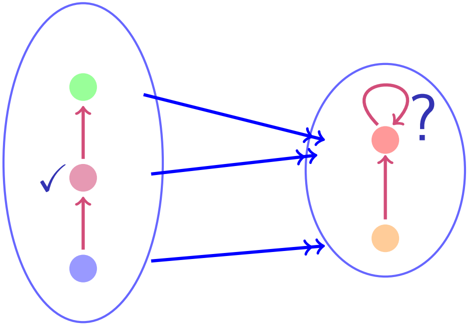

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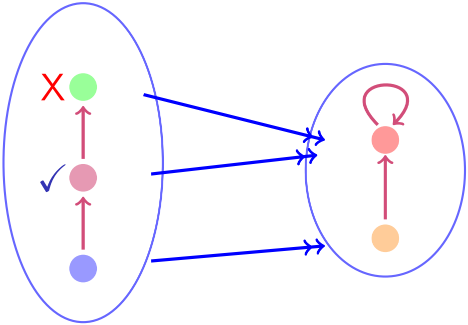

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Bounded morphisms and BAO homomorphisms

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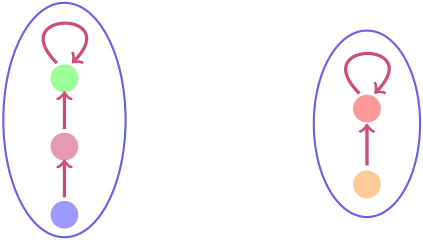

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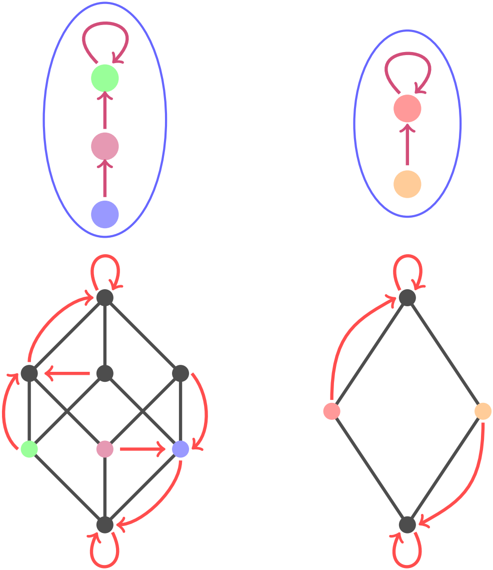

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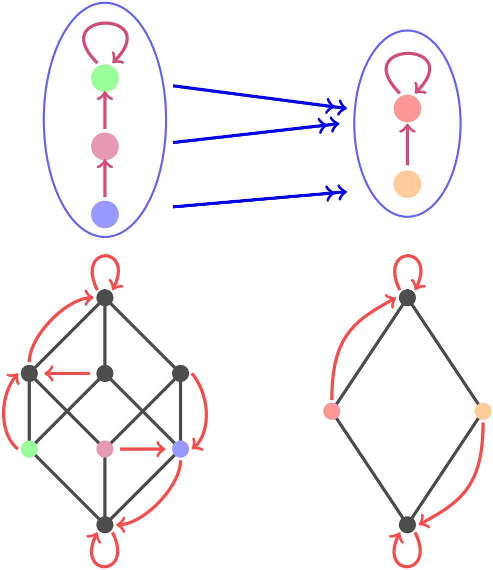

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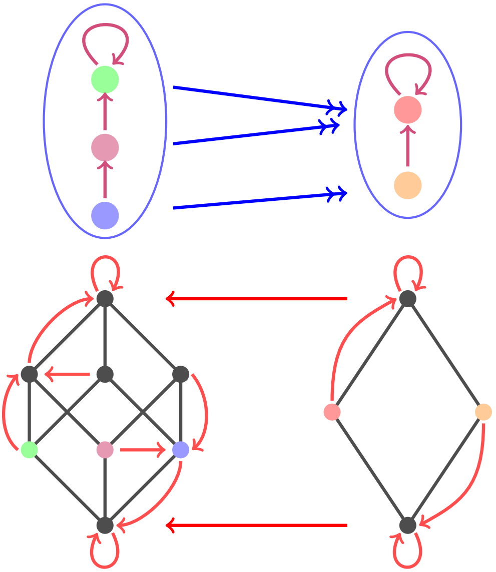

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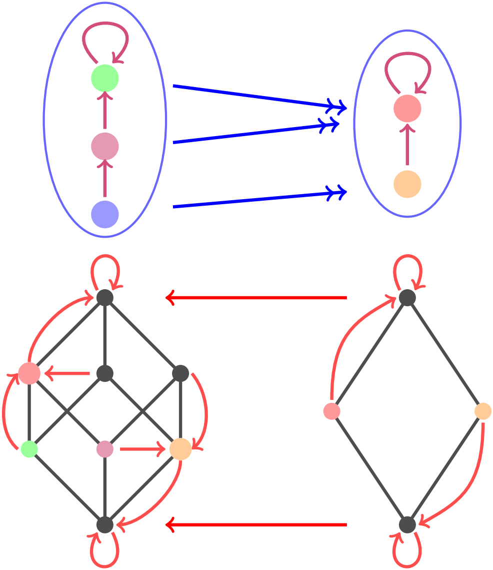

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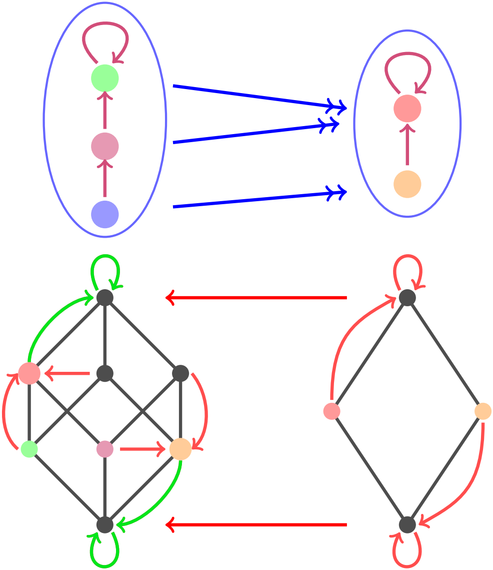

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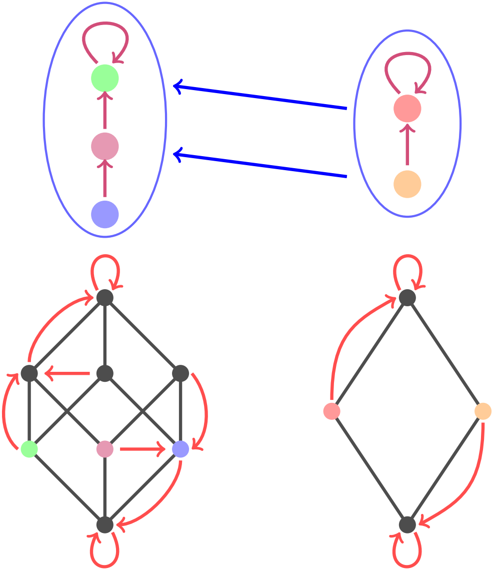

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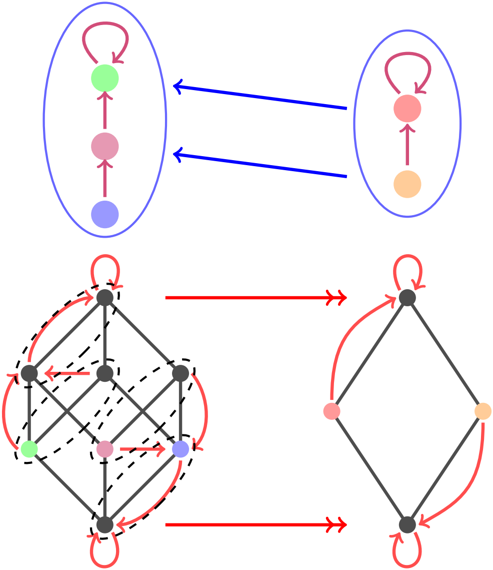

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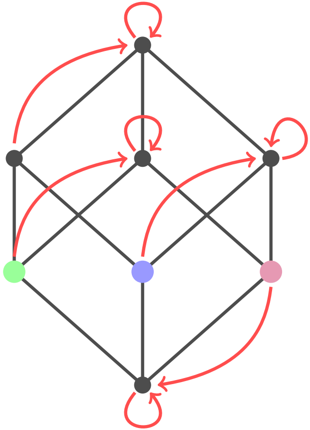

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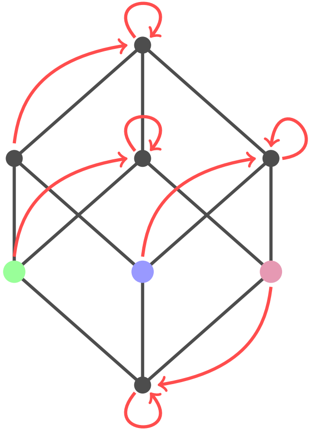

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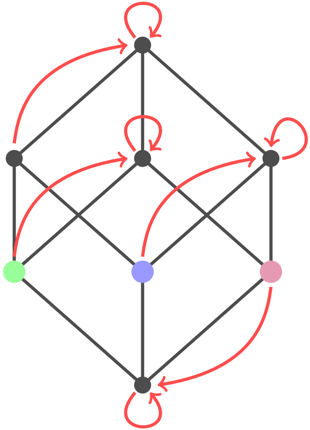

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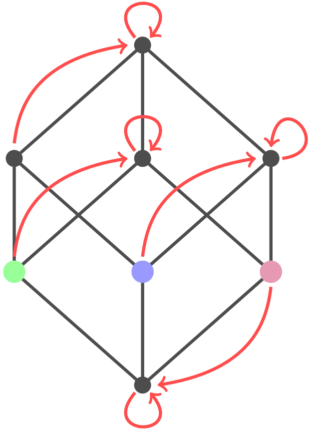

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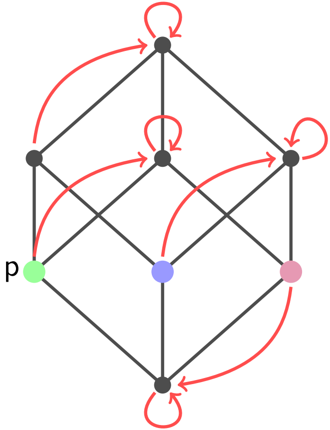

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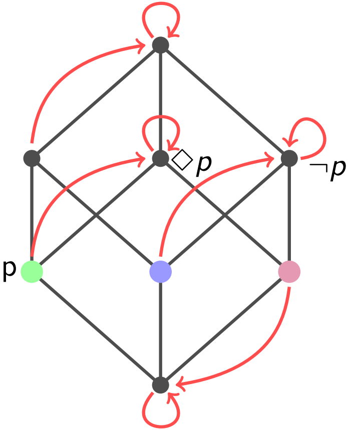

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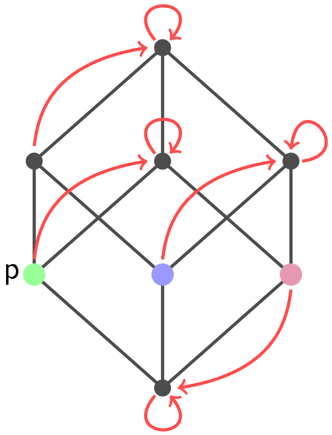

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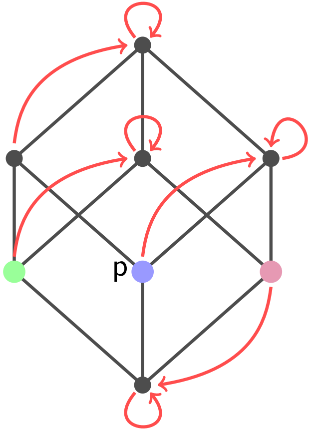

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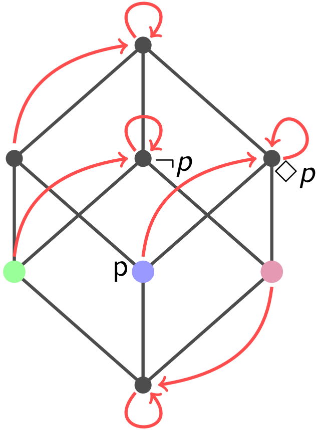

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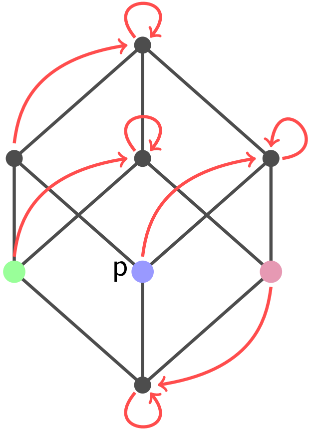

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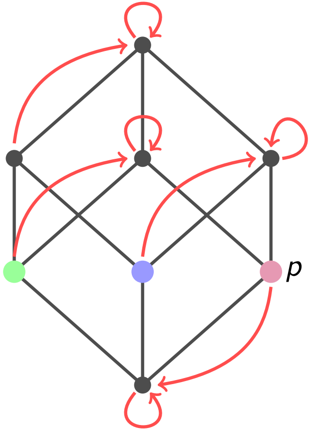

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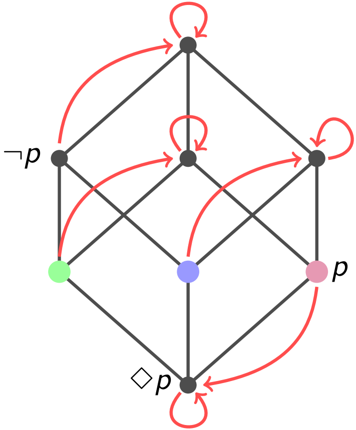

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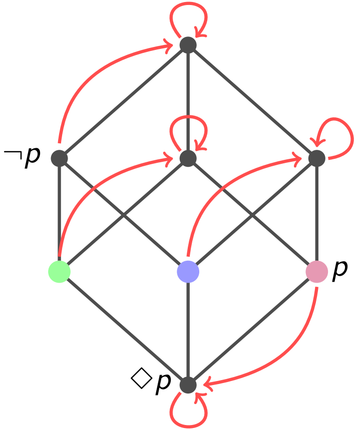

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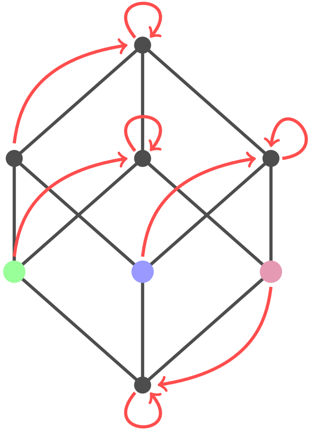

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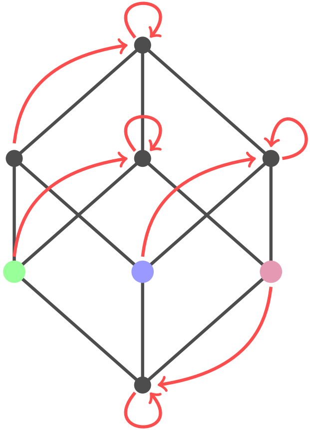

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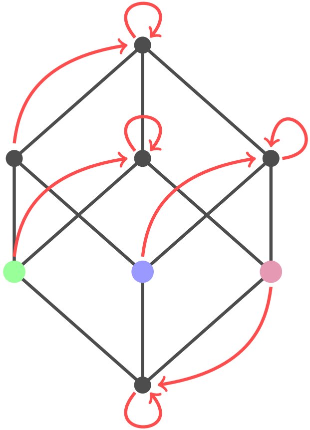

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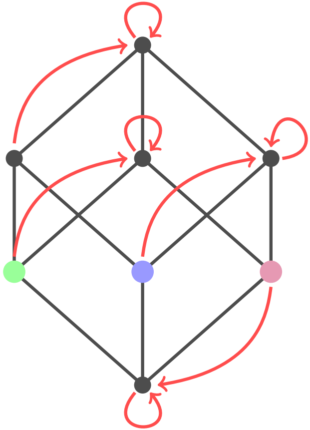

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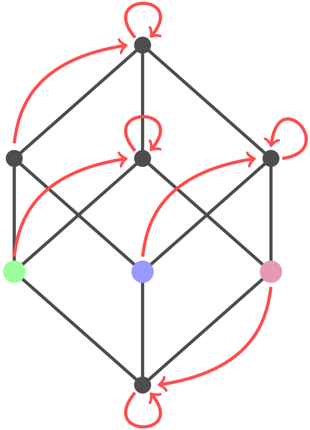

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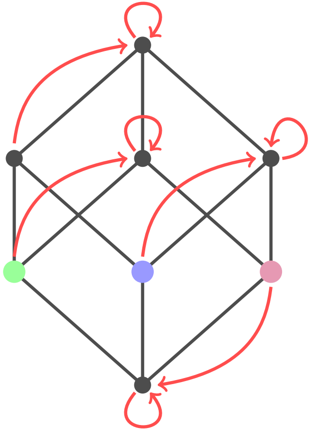

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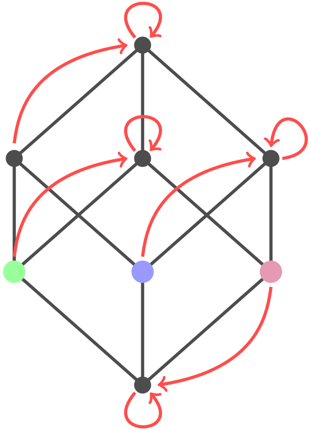

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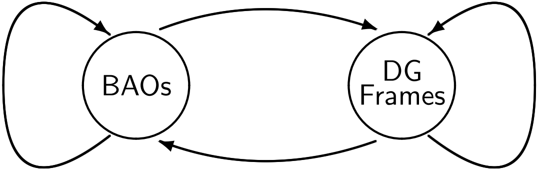

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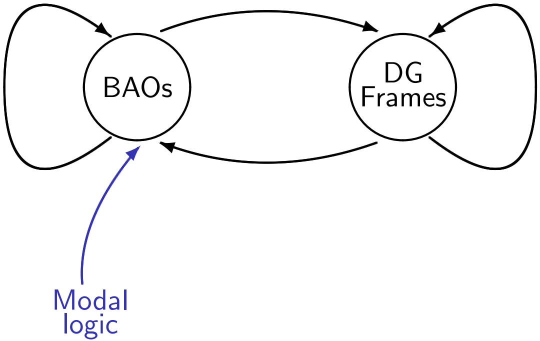

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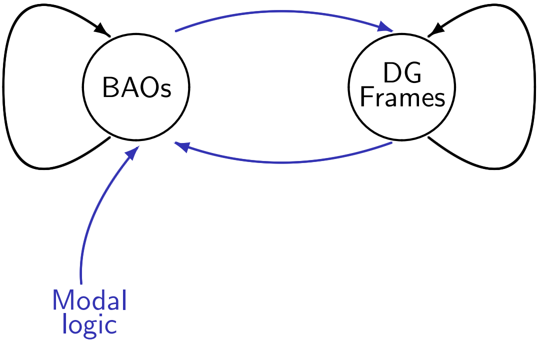

Duality-derived Soundness & Completeness

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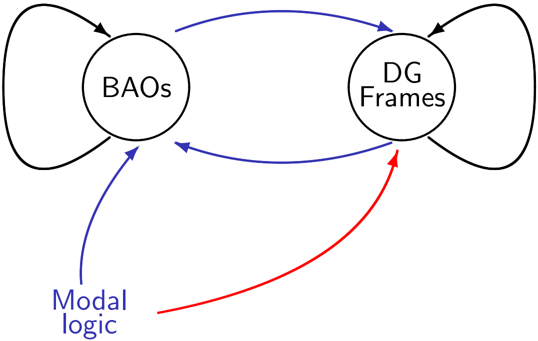
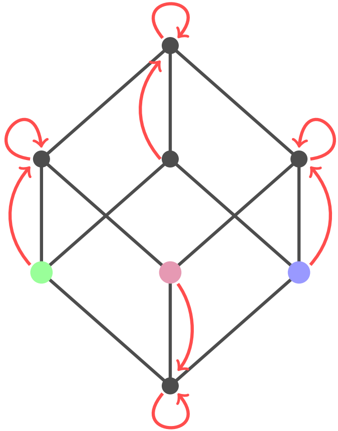


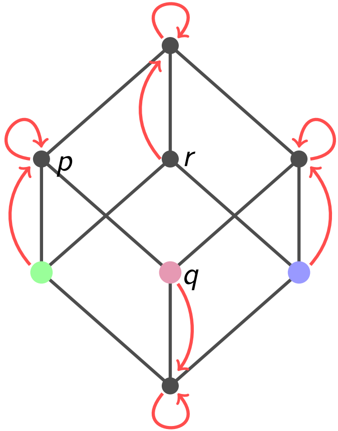
Algebraic s&c + duality $\Rightarrow$ DG-frames s&c

Assignments on algebras $\leftrightsquigarrow$ valuations on frames:

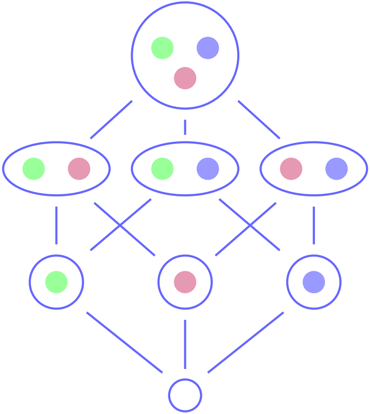
Assignments on algebras ⇝ valuations on frames:

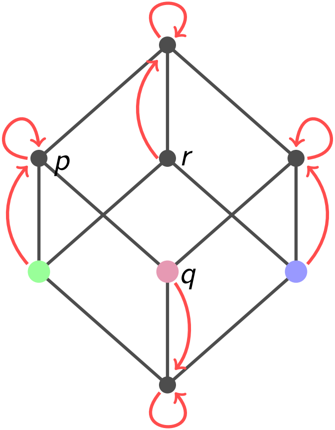


Assignments on algebras ⇝ valuations on frames:

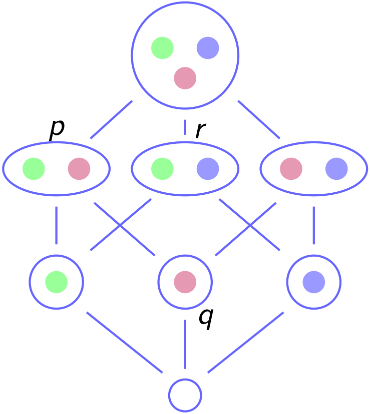


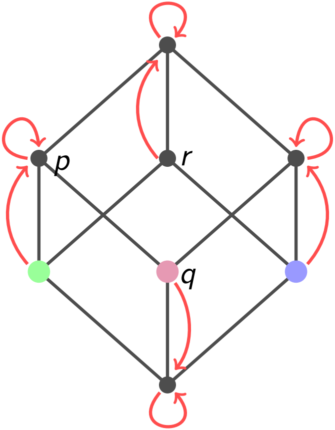
Assignments on algebras ⇜⇝ valuations on frames:



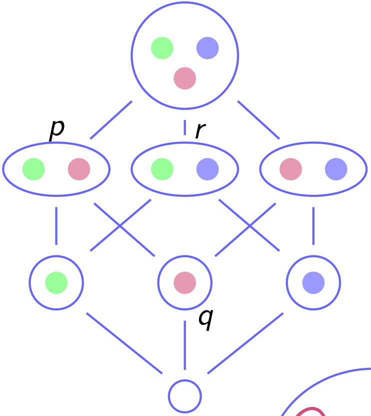


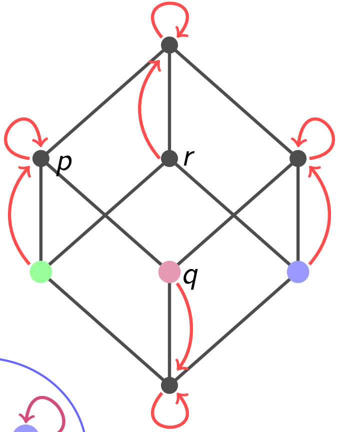
Assignments on algebras ⇝ valuations on frames:

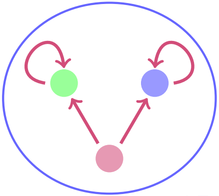




Assignments on algebras $\leftrightsquigarrow$ valuations on frames:







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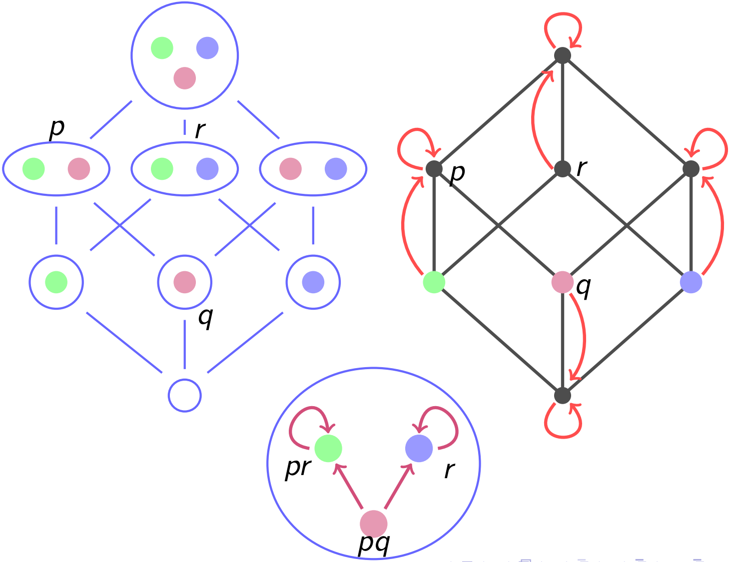
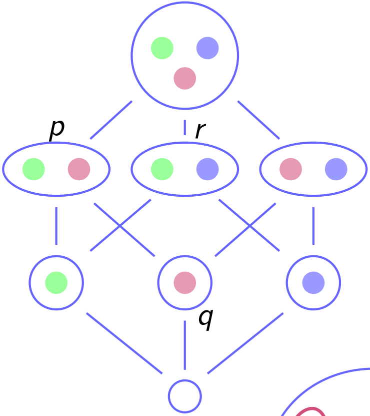
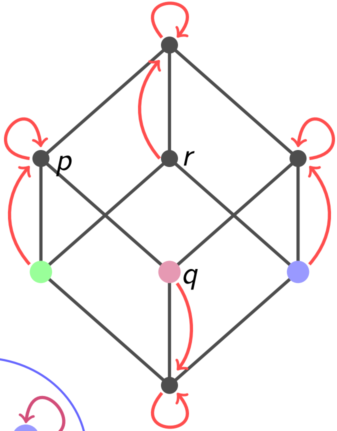
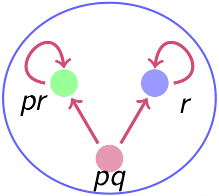


Assignments on algebras $\leftrightsquigarrow$ valuations on frames:





$\mathcal{F}, V, x \Vdash \varphi_1 = \neg r \vee \Diamond p$



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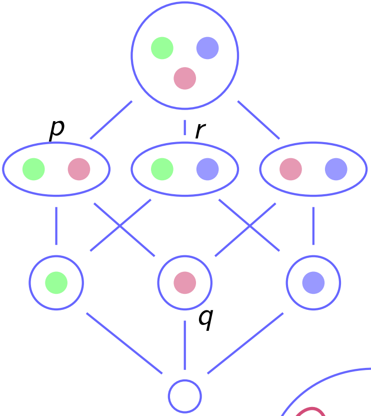

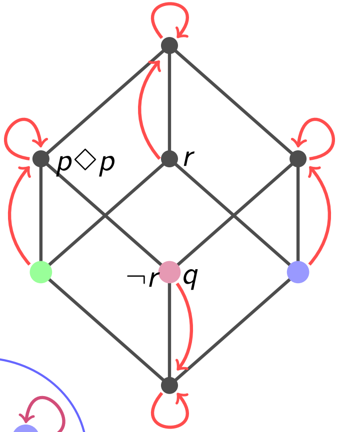

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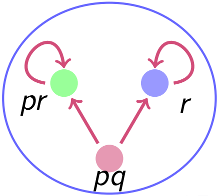

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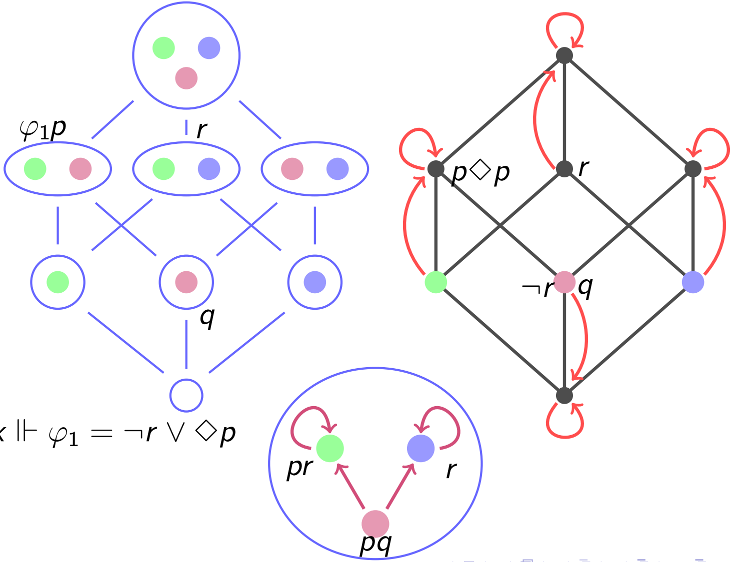

$\mathcal{F}, V, x \Vdash \varphi_1 = \neg r \vee \Diamond p$

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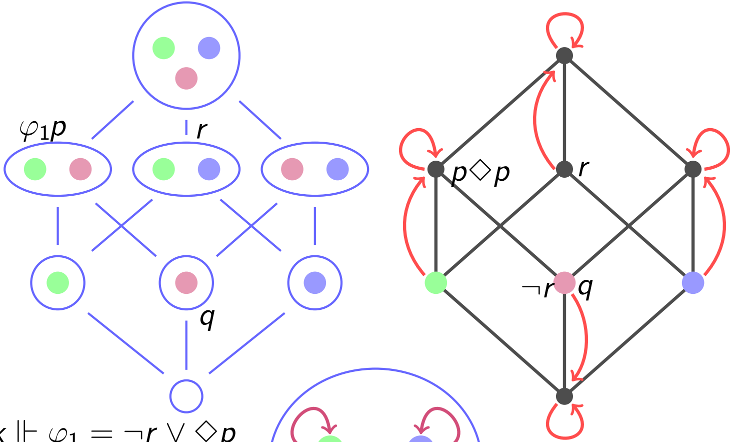
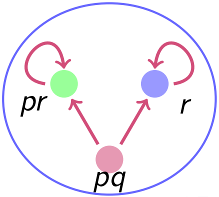
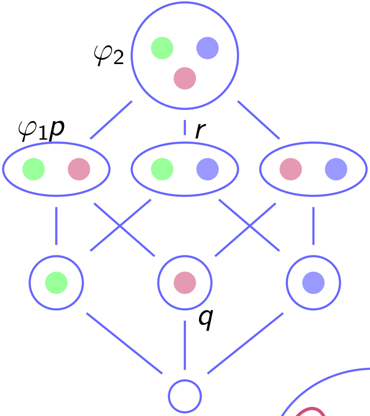
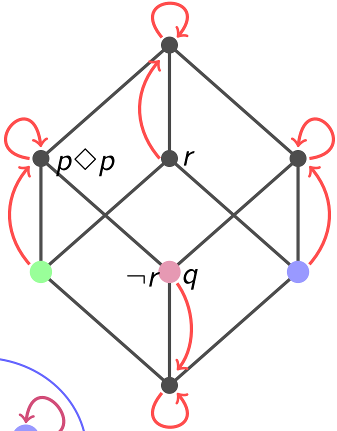


$$\mathcal{F}, V, x \Vdash \varphi_1 = \neg r \vee \Diamond p$$
$$\mathcal{F}, V \Vdash \varphi_2 = r \vee \Diamond p$$



Assignments on algebras $\leftrightsquigarrow$ valuations on frames:





$$\mathcal{F}, V, x \Vdash \varphi_1 = \neg r \vee \Diamond p$$
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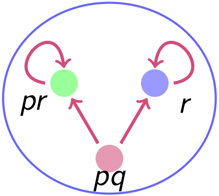

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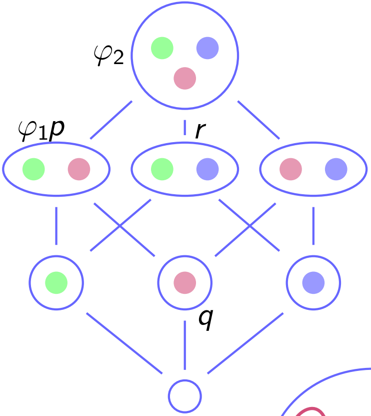
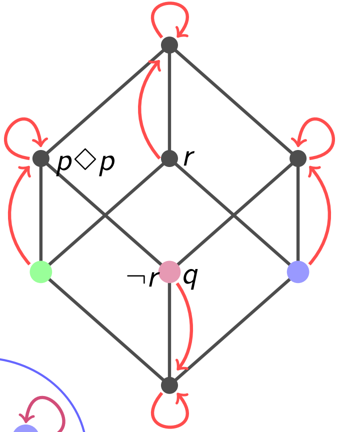
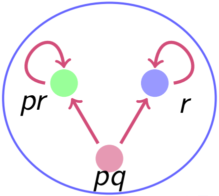
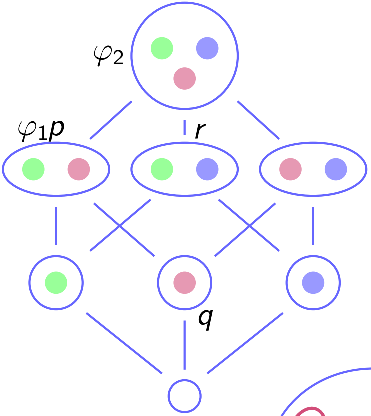


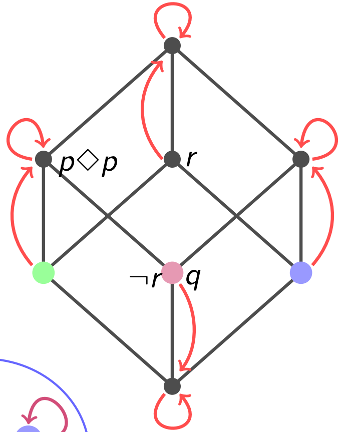


$$\mathcal{F}, V, x \Vdash \varphi_1 = \neg r \vee \Diamond p$$
$$\mathcal{F}, V \Vdash \varphi_2 = r \vee \Diamond p$$
$$\mathcal{F}, x \Vdash \varphi_3 = p \rightarrow \Diamond p$$



Assignments on algebras $\leftrightsquigarrow$ valuations on frames:



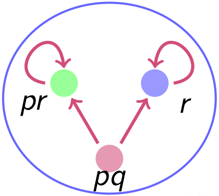


$$\mathcal{F}, V, x \Vdash \varphi_1 = \neg r \vee \Diamond p$$

$$\mathcal{F}, V \Vdash \varphi_2 = r \vee \Diamond p$$

$$\mathcal{F}, x \Vdash \varphi_3 = p \rightarrow \Diamond p$$

$$\mathcal{F} \Vdash \varphi_4 = p \rightarrow \Diamond p$$



Conclusions

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