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Core Logic
Sat Nov 28 18:15:14 CET 2009

Exercise 22

1) G0: total. For any non-bottom element x , either $\neg x$ has to be the bottom element, which is unique, or an element immediately preceding the bottom element which is neither above nor below x , which in the case of G0 is only a_2 and a_3 , so these also have a unique counterpart.

If x is the bottom element, then $\neg x$ is the greatest element if it exists uniquely, which it does.

i.e., $\neg a_2 = a_3$, $\neg a_3 = a_2$, $\neg a_4 = a_0$ and for the rest $\neg x = a_4$

G1: total. Similar to above but simpler because there is only a single element preceding the bottom element.

i.e., $\neg b_6 = b_0$, for the rest $\neg x = b_6$

G2: partial, try $\neg c_1$, then c_2 and c_3 both yield the bottom element but c_2 and c_3 have no path between them, so they don't satisfy the uniqueness condition.

2) No, take for example $x = a_1$. Then $\neg x = a_4$, but $\neg\neg x = a_0$, since a_0 is the greatest element.

3) Yes.

$\neg x$ will always be the bottom element (b_6),

$\neg\neg x$ will always be the greatest element (b_0),

applying the first observation to the second, $\neg\neg\neg x$ will be the bottom element again.

4) A Heyting algebra is a Boolean algebra with only a pseudo-complement instead of a real complement. While $x \wedge \neg x = 0$, it is not generally the case that $x \vee \neg x = 1$ (law of excluded middle). Thus, Heyting algebras are useful for representing intuitionistic logic. G0 is a Heyting algebra because the law of double negation does not hold, and as a consequence, neither the law of excluded middle.

Exercise 23

1) False, he wanted to proceed in incremental steps: "Hilbert thus envisaged his foundational project as a stepwise 'simultaneous development of logic and mathematics,' [...] ". He first wanted to formalize arithmetic, then analysis, and let the rest follow later.

2) Winter 1921: " [...] Hilberts lecture course on Grundlagen der Mathematik of 1921/22 (Hilbert, 1922a, 1922b), where the epsilon-operator is first used [...] " (p. 4)

3) Tau and alpha can be eliminated from any proof by finding numerical substitutions, after which consistency can be shown using the epsilon-substitution method.

4) A shift in emphasis towards primitive recursion, in the hope that it could subsume all number theoretic functions (p. 10)

5) In Hilbert's (1923) system, epsilon A denotes a counterexample, whereas in Ackermann's (1924) system, epsilon A denotes a witness.

6) Ackermann (p. 32)

Exercise 24

- 1) There is the trivial model containing PA and $2+2=5$. This will cause a contradiction with $2+2=4$ as following from PA, and thus, by Ex Falso Sequitur Quodlibet, all statements of arithmetic will follow.
- 2) $2+2=5$
- 3) ?

Exercise 25

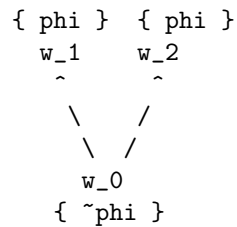
$DN_0 = \Box \sim \Box \sim \text{phi} \rightarrow \text{phi}$

We have that $\sim \Box \sim$ is equivalent with $\langle \rangle$,

so $DN_0 = \Box \langle \rangle \text{phi} \rightarrow \text{phi}$

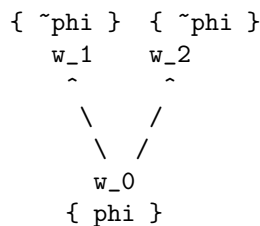
Similarly for DN_1 , $DN_1 = \text{phi} \rightarrow \Box \langle \rangle \text{phi}$

- 1) Neither DN_0 and DN_1 hold in S4-frames. Countermodel for DN_0 :



(Reflexive connections left out). From the point of view of w_0 it is necessarily the case that possibly phi : both in w_1 and in w_2 , due to the reflexive connections, phi is possible. In w_0 as well, phi is possible because it is true in both w_1 and w_2 . The consequent however is not true, because phi is false in w_0 .

By switching the negations we have a countermodel for DN_1 :



The same reasoning applies here in reverse. Although phi is true in w_0 it is not the case that in all worlds it is possible, because w_1 and w_2 are counterexamples to this.

2)

DN₀ is not valid in S5-frames, the first countermodel of the previous question applies here as well, if we make the links to w₁ and w₂ bidirectional.

DN₁ is valid in S5-frames, because it is the formula that characterizes symmetric frames. Proof that DN₁ is valid in symmetric frames:

Suppose R is a symmetric relation on W in F for some w in W:

- suppose phi is false in w, then the conditional is true
- alternatively, suppose phi is true in w, then $\Box \langle \text{phi} \rangle$ must be true as well
- to see that this is the case, consider each world w' such that wRw', in each of these $\langle \text{phi} \rangle$ must hold. Since the relation is symmetric there is always the world w accessible from w', such that $\langle \text{phi} \rangle$ holds.