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Core Logic
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Exercise 22
1) G0: total. For any non-bottom element x, either -x has to be the bottom element, which is unique, or an element immediately preceding the bottom element which is neither above nor below x, which in the case of G0 is only a2 and a3, so these also have a unique counterpart.
If x is the bottom element, then -x is the greatest element if it exists uniquely, which it does.
i.e., -a2 = a3, -a3 = a2, -a4 = a0 and for the rest -x = a4
G1: total. Similar to above but simpler because there is only a single element preceding the bottom element.
i.e., -b6 = b0, for the rest -x = b6
G2: partial, try -c1, then c2 and c3 both yield the bottom element but c2 and c3 have no path between them, so they don't satisfy the uniqueness condition.
2) No, take for example x = a1. Then -x = a4, but --x = a0, since a0 is the greatest element.
3) Yes.
-x will always be the bottom element (b6),
--x will always be the greatest element (b0),
applying the first observation to the second, ---x will be the bottom element again.
4) A Heyting algebra is a Boolean algebra with only a pseudo-complement instead of a real complement. While x ^ -x = 0, it is not generally the case that x v -x = 1 (law of excluded middle). Thus, Heyting algebras are useful for representing intuitionistic logic. G0 is a Heyting algebra because the law of double negation does not hold, and as a consequence, neither the law of excluded middle.

Exercise 23
1) False, he wanted to proceed in incremental steps: ''Hilbert thus envisaged his foundational project as a stepwise 'simultaneous development of logic and mathematics,' [...] ''. He first wanted to formalize arithmetic, then analysis, and let the rest follow later.
2) Winter 1921: '' [...] Hilberts lecture course on Grundlagen der Mathematik of 1921/22 (Hilbert, 1922a, 1922b), where the epsilon-operator is first used [...]'' (p. 4)
3) Tau and alpha can be eliminated from any proof by finding numerical substitutions, after which consistency can be shown using the epsilon-substitution metod.
4) A shift in emphasis towards primitive recursion, in the hope that it could subsume all number theoretic functions (p. 10)
5) In Hilbert's (1923) system, epsilon A denotes a counterexample, whereas in Ackermann's (1924) system, epsilon A denotes a witness.
6) Ackermann (p. 32)

Exercise 24
1) There is the trivial model containing PA and 2+2=5. This will cause a contradiction with 2+2=4 as following from PA, and thus, by Ex Falso Sequitur Quodlibet, all statements of arithmetic will follow.
2) 2+2=5
3) ?

Exercise 25

DN_0 = []~[]~ phi -> phi
We have that ~[]~ is equivalent with <>,
so DN_0 = []<> phi -> phi
Similarly for DN_1, DN_1 = phi -> []<>phi
1) Neither DN_0 and DN_1 hold in S4-frames. Countermodel for DN_0:

```
    { phi }  { phi }
      w_1    w_2
       ^      ^
        \    /
         \  /
         w_0
       { ~phi }
```

(Reflexive connections left out). From the point of view of w_0 it is necessarily the case that possibly phi: both in w_1 and in w_2, due to the reflexive connections, phi is possible. In w_0 as well, phi is possible because it is true in both w_1 and w_2. The consequent however is not true, because phi is false in w_0.

By switching the negations we have a countermodel for DN_1:

```
   { ~phi }  { ~phi }
      w_1    w_2
       ^      ^
        \    /
         \  /
         w_0
       { phi }
```

The same reasoning applies here in reverse. Although phi is true in w_0 it is not the case that in all worlds it is possible, because w_1 and w_2 are counterexamples to this.

2)

DN_0 is not valid in S5-frames, the first countermodel of the previous question applies here as well, if we make the links to w_1 and w_2 bidirectional.

DN_1 is valid in S5-frames, because it is the formula that characterizes symmetric frames. Proof that DN_1 is valid in symmetric frames:

Suppose R is a symmetric relation on W in F for some w in W:

- suppose phi is false in w, then the conditional is true
- alternatively, suppose phi is true in w, then []<>phi must be true as well
- to see that this is the case, consider each world w' such that wRw', in each of these <>phi must hold. Since the relation is symmetric there is always the world w accessible from w', such that <>phi holds.