

Exercise 1

Theorem: if ϕ is a formula in a first-order language, then there is an equivalent formula ϕ' using only implication, negation and existential and universal quantification as logical constants.

Proof: induction on complexity of ϕ

base case: $\phi = \phi'$, since no logical constants occur

induction step:

assume that for ψ and χ we have ψ' and χ' , such that ψ is equivalent

with ψ' and ψ' uses only implication negation and existential quantification, and mutatis mutantibus for χ .

The following cases occur:

- $\phi = \sim\psi$
 $\phi' = \sim\psi'$
- $\phi = \exists x \psi$
 $\phi' = \exists x \psi'$
- $\phi = \forall x \psi$
 $\phi' = \forall x \psi'$
- $\phi = \psi \rightarrow \chi$
 $\phi' = \psi' \rightarrow \chi'$
- $\phi = \psi \wedge \chi$
 $\phi' = \sim(\psi' \rightarrow \sim\chi')$
- $\phi = \psi \vee \chi$
 $\phi' = \sim\psi' \rightarrow \chi'$

The last three equivalences follow from propositional logic.

Lemma:

if x does not occur freely in A

$A \rightarrow \forall x: B \Leftrightarrow \forall x: (A \rightarrow B)$

- | | | |
|----|--------------------------------|-----------------------|
| 1. | $A \rightarrow \forall x: B$ | pr. |
| 2. | +- A | A |
| 3. | $\forall x: B$ | E- \rightarrow 1, 2 |
| 4. | B | EV |
| | +----- | |
| 5. | $A \rightarrow B$ | I- \rightarrow |
| 6. | $\forall x: (A \rightarrow B)$ | IV |

- | | | |
|----|--------------------------------|------------------|
| 1. | $\forall x: (A \rightarrow B)$ | pr. |
| 2. | $A \rightarrow B$ | EV |
| 3. | +- A | A |
| 4. | B | E- \rightarrow |
| 5. | $\forall x: B$ | IV |
| | +----- | |
| 6. | $A \rightarrow \forall x: B$ | I- \rightarrow |

$\exists x: B \rightarrow A \Leftrightarrow \forall x: (B \rightarrow A)$

1.	$\exists x: B \rightarrow A$	pr.
2.	+ - B	A
3.	$\exists x B$	I \exists 2
4.	A	E \rightarrow 1,3
+-----		
5.	B \rightarrow A	I \rightarrow
6.	$\forall x: (B \rightarrow A)$	I \forall

1.	$\forall x: (B \rightarrow A)$	pr.
2.	B \rightarrow A	E \forall
3.	+ - $\exists x: B$	A
4.	A	E \exists 2,3
+-----		
5.	$\exists x: B \rightarrow A$	I \rightarrow

$A \rightarrow \exists x: B \Leftrightarrow \exists x: (A \rightarrow B)$

1.	A \rightarrow $\exists x: B$	pr.
2.	+ - - A	A
3.	$\exists x: B$	E \rightarrow 1,2
4.	+ - B	A
6.	B	rep. 4
+-----		
7.	B \rightarrow B	I \rightarrow
8.	B	E \exists 3,7
+-----		
9.	A \rightarrow B	I \rightarrow
10.	$\exists x (A \rightarrow B)$	I \exists

1.	$\exists x: (A \rightarrow B)$	pr.
2.	+ - - A \rightarrow B	A
3.	+ - A	A
4.	B	E \rightarrow 2,3
5.	$\exists x: B$	I \exists
+-----		
6.	A \rightarrow $\exists x: B$	I \rightarrow
+-----		
7.	(A \rightarrow B) \rightarrow (A \rightarrow $\exists x: B$)	I \rightarrow
8.	A \rightarrow $\exists x: B$	E \exists 1,7

$\forall x: B \rightarrow A \Leftrightarrow \exists x: (B \rightarrow A)$

1.	$\forall x: B \rightarrow A$	pr.
2.	$\sim \exists x: (B \rightarrow A)$	A
3.	[...]	

1.	$\exists x: (B \rightarrow A)$	pr.
2.	+ - - $\forall x: B$	A
3.	B	E \forall
4.	+ - B \rightarrow A	A
5.	A	E \rightarrow 3,4
+-----		
6.	(B \rightarrow A) \rightarrow A	I \rightarrow
7.	A	E \exists 1,6
+-----		
8.	$\forall x: B \rightarrow A$	I \rightarrow

By the completeness theorem we know that these derivations show that these equivalences hold.

Exercise 2

i) Given the Completeness theorem Δ has a model. Now, consider an arbitrary model $M = D, I$ of Δ . We construct a model $M' = D', I'$ such that M' is a restriction of M .

D' is a subset of D , such that for every d in D' and d' in D' , either $d = d'$, or there is some P such that d is in $I(P)$ whereas d' is not in $I(P)$, or vice versa. In other words, the objects in D' are discernable by at least one predicate.

I' is defined thus: For every d in D' , $I'(d) = I(d)$

Because we have a finite number of predicates, there are only a finite number of distinct objects required. In other words, if we have n predicates, then each object in our domain can be characterized by a n -digit bit string, of which there are only a finite number possible. Hence the restricted model is finite.

ii) Given the previous theorem we can decide for any formula in the language L whether it is a logical validity. We do this by enumerating all possible models, starting by interpreting each predicate occurring in the formula as the empty set, and then adding objects step-by-step. After a finite amount of steps we reach the interpretation where each predicate is interpreted with the whole domain, because both the domain and number of predicates is finite. If we have not encountered an interpretation where the formula is false, then the formula is a logical validity.

Exercise 3

i)

Given that ϕ is valid in all infinite structures, ϕ cannot specify anything in particular, so the sentence ϕ must express that there are at least n objects in the domain, for an arbitrary n .

Consider a sentence ϕ such that:

$\phi = \exists x_1 \dots \exists x_n: x_1 \neq x_2 \dots x_1 \neq x_n \dots x_2 \neq x_3 \dots x_{n-1} \neq x_n$

All infinite structures model ϕ , but for every value of n we can construct a model with $n+1$ objects which will also be a model of ϕ .

ii)

We can imagine Σ as consisting of the following infinite set:

$\phi_2 = \exists x_1 \exists x_2: x_1 \neq x_2$

$\phi_3 = \exists x_1 \exists x_2 \exists x_3: x_1 \neq x_2 \wedge x_1 \neq x_3 \wedge x_2 \neq x_3$

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Now by the compactness theorem it may seem that this theory has a finite model, but in fact it is not possible to find an n such that it is big enough to satisfy all the formulas in Σ , because for every finite n , we simply present the sentence ϕ_{n+1} to contradict the claim that a model with n elements could satisfy Σ .