0440949 Andreas van Cranenburgh. Basic Logic homework 3.

Exercise 11.

Step f) At step n+1 I^ is applied. In this case there are
steps i <= n and j <= n, such that:
i) At step i a sentence ψ has been introduced. At step j a
sentence χ has been introduced. Sentence φ = ψ ^ χ.
ii) the union of Gamma_i and Gamma_j is a proper subset of
Gamma_n+1
Following the induction hypothesis we may assume that:
the union of Delta and Gamma_i entails ψ
the union of Delta and Gamma_j entails χ
from this it follows, using the semantic definition of
conjunction, that the union of Delta and Gamma_n+1 entails φ.

Step g) At step n+1 E^ is applied. In this case there is
step i <= n, such that:
i) At step i a sentence ψ has been introduced, of the form
ψ = φ ^ χ
or ψ = χ ^ φ.
ii) Gamma_i is a proper subset of Gamma_n+1
Following the induction hypothesis we may assume that:
the union of Delta and Gamma_i entails ψ
from this it follows, using the semantic definition of
conjunction, that the union of Delta and Gamma_n+1 entails φ.

Step h) At step n+1 Iv is applied. In this case there is
step i <= n, such that:
i) At step i a sentence ψ has been introduced. There is an
arbitrary sentence χ. Sentence φ = ψ v χ.
ii) Gamma_i is a proper subset of Gamma_n+1
Following the induction hypothesis we may assume that:
the union of Delta and Gamma_i entails ψ
from this it follows, using the semantic definition of
disjunction, that the union of Delta and Gamma_n+1 entails φ.

Step j) Application of E->
Step k) Application of I->
Step l) Application of E~~

Exercise 12
Suppose ~~φ is derivable from Delta in intuitionistic logic, then
there is a simple derivation in classical logic to obtain φ:

.
.
.
n ~~φ
n+1 φ E~~

Converse:
Suppose there is a derivation of φ from Delta in classical logic.

base case: At step 1 we have the following possibilities:
- a premise is introduced
- an assumption is made
Both of these steps can be made in intuitionistic logic as well.

induction step:
induction hypothesis: if φ_n can be derived in classical logic, then ~~φ_n can be derived in intuitionistic logic.
Suppose the induction hypothesis holds, then at the next step the following possibilities can occur:
a) a premise is introduced
b) an assumption is made
c) ⊥ is introduced
d) Negation introduction
these four can be made in intuitionistic logic as well so they don't contradict the induction hypothesis.

e) EFSQ
At some step i <= n we have ⊥, under the same assumptions as step n.
we assume ~φ_n+1
repeat ⊥
withdraw assumption, introduce negation. No we have ~~φ_n+1.
f) Conjunction introduction, see lemma (ii)
g) conjuction elimination, see lemma (iii)
h) disjunction introduction, see lemma (iv)
i) disjunction elimination, see lemma (v)
j) Application of E->
For some i and j >= n, φ_i = ~~φ and φ_j = ~~(φ -> ψ). By lemma (vii) ~~ψ can be derived.
k) Application of I->, at step n some φ is in Gamma and some formula ~~ψ has been derived. Then by lemma (vi), ~~(φ -> ψ) can be derived. Alternatively, the formula ψ has been derived from φ. In this case φ -> ψ can simply be derived.
l) Application of E~~
If φ_n has an even number of negations greater than 2, then lemma (viii) is applied. Otherwise this step is skipped.

Lemma:

(iii)

```
1.     ~~(φ ^ ψ)   pr.
2. +-- ~φ          A
3. | - φ ^ ψ       A
4. | | φ           E^3
5. | | ⊥           E~2,4
6. | | ~(φ ^ ψ)    I~
   | +--------------------
7. |   ⊥           E~1,6
   +----------------------
8.     ~~φ         I~
```

(vi)
still looking for this one...

(vii)

```
1.        ~~φ       pr.
2.        ~~(φ -> ψ)      pr.
3. +----- ~ψ        A
4. | +-- φ -> ψ     A
5. | | - φ         A
6. | | | ψ         E->4,5
7. | | | ⊥         E~3,6
   | | +-------------------
8. | |   ~φ        I~
9. | |   ⊥         E~1,8
   | +---------------------
10.|     ~(φ->ψ)    I~
11.|     ⊥          E~10,2
   +------------------------
12.       ~~ψ       I~
```

**Exercise 13**

```
1. p -> q                              prem.
2. q -> r                              prem.
3. (q -> r) -> (p -> (q ->r ))                  Axiom i
4. p -> (q -> r)                       2,3 mp
5. (p -> (q -> r)) -> ((p -> q) -> (p -> r))    Axiom ii
6. (p -> q) -> (p -> r)                      4,5 mp
7. p -> r                              1,6 mp
```

**Exercise 14**

By the deduction theorem it is sufficient to show that
p -> (q -> r), q  |- p -> r

```
1. p -> (q -> r)                       prem.
2. q                                   prem.
3. q -> (p -> q)                       Axiom i
4. p -> q                              2,3 mp
5. (p -> (q -> r)) -> ((p -> q) -> (p -> r))    Axiom ii
6. (p -> q) -> (p -> r)                      4,5 mp
7. p -> r                              4,6 mp
```

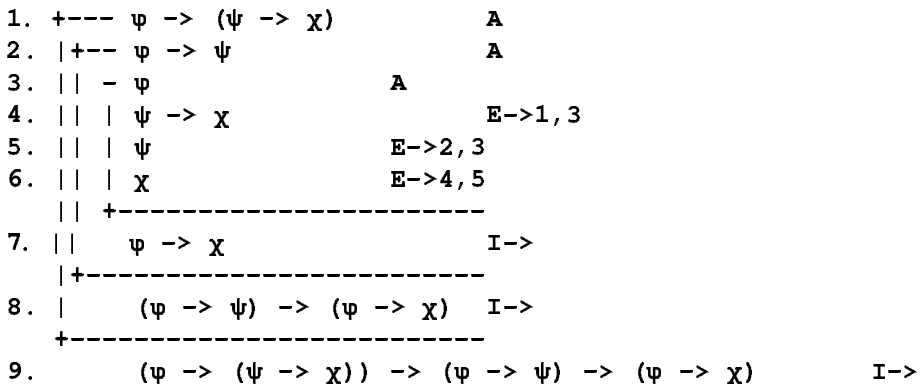
**Exercise 15**

Lemma:
Axioms derived using natural deduction:

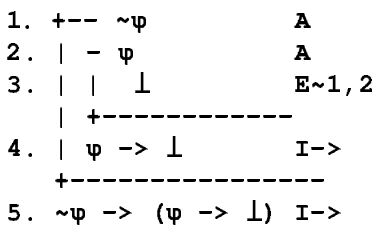
(i)

```
1. +-- φ           A
2. | - ψ           A
3. | | φ           rep 1
   | +-----------------------
4. |   ψ -> φ            I->
   +-------------------------
5.     φ -> (ψ -> φ)     I->
```

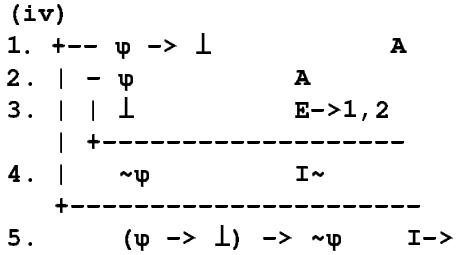
(ii)


1. +--- φ -> (ψ -> χ) A
2. |+-- φ -> ψ A
3. || - φ A
4. || | ψ -> χ E->1,3
5. || | ψ E->2,3
6. || | χ E->4,5
|| +----------
7. || φ -> χ I->
|+----------
8. | (φ -> ψ) -> (φ -> χ) I->
+----------
9. (φ -> (ψ -> χ)) -> (φ -> ψ) -> (φ -> χ) I->

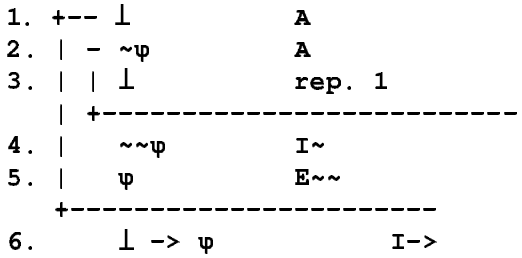
(iii)


1. +-- ~φ A
2. | - φ A
3. | | ⊥ E~1,2
| +----------
4. | φ -> ⊥ I->
+----------
5. ~φ -> (φ -> ⊥) I->

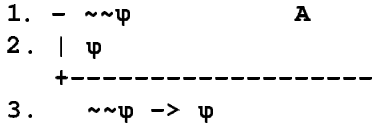
(iv)


1. +-- φ -> ⊥ A
2. | - φ A
3. | | ⊥ E->1,2
| +----------
4. | ~φ I~
+----------
5. (φ -> ⊥) -> ~φ I->

(v)


1. +-- ⊥ A
2. | - ~φ A
3. | | ⊥ rep. 1
| +----------
4. | ~~φ I~
5. | φ E~~
+----------
6. ⊥ -> φ I->

(vi)


1. - ~~φ A
2. | φ E~~
+----------
3. ~~φ -> φ I->

Suppose there is an axiomatic deduction of φ from Delta, then there is also a natural deduction. Let φ_k be the sentence derived at step k. We prove by induction on k that φ_k is derivable using natural deduction:

base case:
a) φ_1 is an element of Delta
b) φ_1 is an axiom
discussed in step case

step case: n = k + 1.
induction hypothesis: Assume there is a natural deduction up till k.
At step n the sentence φ_n is derived in one of the following ways:
a) φ_n is an element of Delta: introduce premise.
b) φ_n is an axiom: introduce derivation of axiom, see lemma.
c) φ_n is obtained using modus ponens, using θ -> φ_n and θ at previous steps. Apply implication elimination.

Suppose there is a natural deduction of φ from Delta, then there is also an axiomatic deduction. Let φ_k be the sentence derived at step k. We prove by induction on k that φ_k is derivable using axiomatic deduction:

base case:
Gamma_0 = {}
a) a premise is introduced
b) an assumption is made
discussed in step case
step case:
induction hypothesis: n = k + 1. Assume there is an axiomatic deduction up till k, with Gamma_k = Gamma_k-1 unless otherwise specified.
At step n the following can happen:
a) a premise is introduced: same in axiomatic deduction
b) an assumption is made: add φ_n to Gamma_n.
c) ⊥ is introduced:
   For some i and j >= n, φ_i = ~φ and φ_j = φ. Derive ψ using modus ponens on axiom (iii) with i and then j.
   1. ~φ -> (φ -> ⊥)  axiom (iii)
   2. φ -> ⊥  mp 1,i
   3. ⊥  mp 2,j
d) Negation introduction: use axiom (v)
e) EFSQ: use axiom (v)
f) Application of E~~, use axiom (vi)
g) Application of E->
   For some i and j >= n, φ_i = φ and φ_j = φ -> ψ. Derive ψ using modus ponens on i and j.
h) Application of I->, at step n-1 some ψ is in Gamma_n-1 and some formula χ derived at step j <= n. Deduction theorem shows that ψ -> χ can be obtained if Delta and ψ entail χ. ψ is removed from Gamma_n

**Exercise 16**

**Conjunction introduction**
**Aiom:** ψ -> (φ -> (φ ^ ψ))
**Conjuction elimination:**
**Axiom:** (φ ^ ψ) -> φ
**Axiom:** (φ ^ ψ) -> ψ
**Disjunction introduction:**
**Axiom:** (φ v ψ) -> φ
**Axiom:** (φ v ψ) -> ψ
**Disjunction elimination:**
**Axiom:** (φ v ψ) -> ((φ -> χ) -> ((φ -> χ) -> χ))