0440949 Andreas van Cranenburgh. Basic Logic homework 3.

Exercise 11.

Step f) At step n+1 I^ is applied. In this case there are
steps i <= n and j <= n, such that:

i) At step i a sentence ψ has been introduced. At step j a sentence χ has been introduced. Sentence $\varphi = \psi \wedge \chi$.

ii) the union of Gamma_i and Gamma_j is a proper subset of Gamma_n+1

Following the induction hypothesis we may assume that:

the union of Delta and Gamma_i entails ψ

the union of Delta and Gamma_j entails χ

from this it follows, using the semantic definition of conjunction, that the union of Delta and Gamma_n+1 entails φ.

Step g) At step n+1 E^ is applied. In this case there is
step i <= n, such that:

i) At step i a sentence ψ has been introduced, of the form

$\psi = \varphi \wedge \chi$

or $\psi = \chi \wedge \varphi$.

ii) Gamma_i is a proper subset of Gamma_n+1

Following the induction hypothesis we may assume that:

the union of Delta and Gamma_i entails ψ

from this it follows, using the semantic definition of conjunction, that the union of Delta and Gamma_n+1 entails φ.

Step h) At step n+1 Iv is applied. In this case there is
step i <= n, such that:

i) At step i a sentence ψ has been introduced. There is an arbitrary sentence χ. Sentence $\varphi = \psi \vee \chi$.

ii) Gamma_i is a proper subset of Gamma_n+1

Following the induction hypothesis we may assume that:

the union of Delta and Gamma_i entails ψ

from this it follows, using the semantic definition of disjunction, that the union of Delta and Gamma_n+1 entails φ.

Step j) Application of E->
Step k) Application of I->
Step l) Application of E~~

Exercise 12

Suppose ~~φ is derivable from Delta in intuitionistic logic, then there is a simple derivation in classical logic to obtain φ:

	.	
	.	
	.	
n	~~φ	
n+1	φ	E~~

Converse:
Suppose there is a derivation of φ from Delta in classical logic.

base case: At step 1 we have the following possibilities:
- a premise is introduced
- an assumption is made
Both of these steps can be made in intuitionistic logic as well.

induction step:
induction hypothesis: if φ_n can be derived in classical logic, then $\sim\sim\varphi_n$ can be derived in intuitionistic logic.
Suppose the induction hypothesis holds, then at the next step the following possibilities can occur:
a) a premise is introduced
b) an assumption is made
c) ⊥ is introduced
d) Negation introduction
these four can be made in intuitionistic logic as well so they don't contradict the induction hypothesis.

e) EFSQ
At some step i <= n we have ⊥, under the same assumptions as step n.
we assume $\sim\varphi_{n+1}$
repeat ⊥
withdraw assumption, introduce negation. No we have $\sim\sim\varphi_{n+1}$.
f) Conjunction introduction, see lemma (ii)
g) conjuction elimination, see lemma (iii)
h) disjunction introduction, see lemma (iv)
i) disjunction elimination, see lemma (v)
j) Application of E->
For some i and j >= n, φ_i = ~~φ and φ_j = ~~(φ -> ψ). By lemma (vii) ~~ψ can be derived.
k) Application of I->, at step n some φ is in Gamma and some formula ~~ψ has been derived. Then by lemma (vi), ~~(φ -> ψ) can be derived.
Alternatively, the formula ψ has been derived from φ. In this case φ -> ψ can simply be derived.
l) Application of E~~
If φ_n has an even number of negations greater than 2, then lemma (viii) is applied. Otherwise this step is skipped.

Lemma:

(iii)
```
1.      ~~(φ ^ ψ)   pr.
2. +-- ~φ           A
3. | - φ ^ ψ        A
4. | | φ            E^3
5. | | ⊥            E~2,4
6. | | ~(φ ^ ψ)     I~
   | +--------------------
7. |   ⊥            E~1,6
   +---------------------
8.      ~~φ         I~
```

(vi)
still looking for this one...

```
(vii)
1.       ~~φ       pr.
2.       ~~(φ -> ψ)      pr.
3. +---- ~ψ        A
4. | +-- φ -> ψ    A
5. | | - φ         A
6. | | | ψ         E->4,5
7. | | | ⊥         E~3,6
   | | +-----------------
8. | |   ~φ        I~
9. | |   ⊥         E~1,8
   | +-------------------
10.|     ~(φ->ψ)   I~
11.|     ⊥         E~10,2
   +---------------------
12.      ~~ψ       I~
```

Exercise 13

```
1. p -> q                              prem.
2. q -> r                              prem.
3. (q -> r) -> (p -> (q ->r ))                      Axiom i
4. p -> (q -> r)                       2,3 mp
5. (p -> (q -> r)) -> ((p -> q) -> (p -> r))    Axiom ii
6. (p -> q) -> (p -> r)                      4,5 mp
7. p -> r                              1,6 mp
```

Exercise 14

By the deduction theorem it is sufficient to show that
p -> (q -> r), q |- p -> r

```
1. p -> (q -> r)                       prem.
2. q                                   prem.
3. q -> (p -> q)                       Axiom i
4. p -> q                              2,3 mp
5. (p -> (q -> r)) -> ((p -> q) -> (p -> r))    Axiom ii
6. (p -> q) -> (p -> r)                      4,5 mp
7. p -> r                              4,6 mp
```

Exercise 15

Lemma:
Axioms derived using natural deduction:

```
(i)
1. +-- φ           A
2. | - ψ           A
3. | | φ           rep 1
   | +-------------------
4. |   ψ -> φ            I->
   +---------------------
5.     φ -> (ψ -> φ)     I->
```

```
(ii)
1. +--- φ -> (ψ -> χ)          A
2. |+-- φ -> ψ                 A
3. || - φ                A
4. || | ψ -> χ                 E->1,3
5. || | ψ                E->2,3
6. || | χ                E->4,5
   || +----------------------
7. ||   φ -> χ                 I->
   |+------------------------
8. |     (φ -> ψ) -> (φ -> χ)  I->
   +-------------------------
9.       (φ -> (ψ -> χ)) -> (φ -> ψ) -> (φ -> χ)        I->

(iii)
1. +-- ~φ          A
2. | - φ           A
3. | |  ⊥          E~1,2
   | +------------
4. | φ -> ⊥        I->
   +---------------
5. ~φ -> (φ -> ⊥) I->

(iv)
1. +-- φ -> ⊥            A
2. | - φ           A
3. | | ⊥           E->1,2
   | +-------------------
4. |    ~φ         I~
   +---------------------
5.      (φ -> ⊥) -> ~φ    I->

(v)
1. +-- ⊥           A
2. | - ~φ          A
3. | | ⊥           rep. 1
   | +-------------------------
4. |    ~~φ        I~
5. |    φ          E~~
   +----------------------
6.      ⊥ -> φ           I->

(vi)
1. - ~~φ           A
2. | φ                   E~~
   +----------------------
3.   ~~φ -> φ            I->
```

Suppose there is an axiomatic deduction of φ from Delta, then there is also a natural deduction. Let φ_k be the sentence derived at step k. We prove by induction on k that φ_k is derivable using natural deduction:

base case:
a) φ_1 is an element of Delta
b) φ_1 is an axiom
discussed in step case

step case: $n = k + 1$.
induction hypothesis: Assume there is a natural deduction up till k.
At step n the sentence φ_n is derived in one of the following ways:
a) φ_n is an element of Delta: introduce premise.
b) φ_n is an axiom: introduce derivation of axiom, see lemma.
c) φ_n is obtained using modus ponens, using $\theta \rightarrow \varphi_n$ and θ at previous steps. Apply implication elimination.

Suppose there is a natural deduction of φ from Delta, then there is also an axiomatic deduction. Let φ_k be the sentence derived at step k. We prove by induction on k that φ_k is derivable using axiomatic deduction:

base case:
$\Gamma_0 = \{\}$
a) a premise is introduced
b) an assumption is made
discussed in step case
step case:
induction hypothesis: $n = k + 1$. Assume there is an axiomatic deduction up till k, with $\Gamma_k = \Gamma_{k-1}$ unless otherwise specified.
At step n the following can happen:
a) a premise is introduced: same in axiomatic deduction
b) an assumption is made: add φ_n to Γ_n.
c) $\bot$ is introduced:
   For some i and $j >= n$, $\varphi_i = \sim\varphi$ and $\varphi_j = \varphi$. Derive ψ using modus ponens on axiom (iii) with i and then j.
   1. $\sim\varphi \rightarrow (\varphi \rightarrow \bot)$ axiom (iii)
   2. $\varphi \rightarrow \bot$ mp 1,i
   3. $\bot$ mp 2,j
d) Negation introduction: use axiom (v)
e) EFSQ: use axiom (v)
f) Application of E~~, use axiom (vi)
g) Application of E->
   For some i and $j >= n$, $\varphi_i = \varphi$ and $\varphi_j = \varphi \rightarrow \psi$. Derive ψ using modus ponens on i and j.
h) Application of I->, at step n-1 some ψ is in Γ_{n-1} and some formula χ derived at step $j <= n$. Deduction theorem shows that $\psi \rightarrow \chi$ can be obtained if Delta and ψ entail χ. ψ is removed from Γ_n

Exercise 16

Conjunction introduction

Aiom: ψ -> (φ -> (φ ^ ψ))

Conjuction elimination:

Axiom: (φ ^ ψ) -> φ

Axiom: (φ ^ ψ) -> ψ

Disjunction introduction:

Axiom: (φ v ψ) -> φ

Axiom: (φ v ψ) -> ψ

Disjunction elimination:

Axiom: (φ v ψ) -> ((φ -> χ) -> ((φ -> χ) -> χ))