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Basic Logic, final exam
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I worked alone.

Exercise 1

We have a derivation of $P(s) \leftrightarrow P(t)$ from Δ, for an arbitrary P, not occurring in Δ or in an open assumption. Since P is arbitrary, we can consider the case where the interpretation of P is the most specific and discriminating one possible: when it is only true for a single object, say the object referred to by s. In this case $P(x)$ is equivalent to saying $s = x$.

With this in mind, we obtain a new derivation by replacing every occurrence of $P(x)$ by $s = x$, and we can proceed with the following steps:

m.	$[P(x)/s = x]P(s) \leftrightarrow P(t)$	
m+1.	$s = s \rightarrow s = t \wedge s = t \rightarrow s = s$	
m+2.	$s = s \rightarrow s = t$	$E \wedge m + 1$
m+3.	$s = s$	$I =$
n.	$s = t$	$E \rightarrow m + 2, m + 3$

If there are variables in s it might be possible to derive unwanted results by introducing quantifiers. To avoid this I suggest reformulating $I ='$ as follows:

1. $P(s)$ assumption

⋮

2. $P(t)$

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3. $s = t$ I='

Exercise 2

(i) $\exists y Rxy$

This formula determines $\{1, 4\}$ because 1 and 4 are the only two assignments for x that make the formula hold in the given interpretation.

(ii) The set $\{2\}$ cannot be determined because the relation R is completely symmetric with respect to the objects 2 and 3, so any formula that determines 2 will also determine 3, and vice versa.

Theorem: a set A can be determined by a two-place predicate if A is the set of nth-arguments for which R holds, or the union or intersection of two such sets.

Proof: If A is the set of first arguments for which R holds, then A contains all elements $d \in D$ such that $\mathcal{M} \models \exists y Rxy \; v(x/d)$, which is {1, 4}. Likewise for second arguments for which R holds, there is the formula $\exists y Ryx$, which determines {2, 3}.

With these two as base cases, we can inductively determine some other sets:

- $\exists y Rxy \wedge \exists y Ryx$ determines the intersection of {1, 4} and {2, 3}, which is the empty set.
- $\exists y Rxy \vee \exists y Ryx$ determines the union of {1, 4} and {2, 3}, which is the whole domain.
- $\neg \exists y Rxy$ or $\exists y \neg Rxy$ determine the same set as $\exists y Ryx$. Negation determines the complement, i.e., using negation will turn {1, 4} into {2, 3} and vice versa, or turn the empty set into the whole domain and vice versa.
- No other sets can be formed because other formulas will arrive at one of the four sets already determined. Changing the quantifier makes no difference, because of the symmetry (both of the objects occurring as first argument point to the same objects occurring as second argument, and vice versa).

(iii) The following subsets are determined by some formula:

- $\varnothing$: Rxx
- {1, 4}: $\exists y Rxy$
- {2, 3}: $\exists y Ryx$
- {1, 2, 3, 4}: $\exists y Rxy \vee \exists y Ryx$

Exercise 3

(i) Suppose $\mathcal{M}$ is an elementary submodel of $\mathcal{M}'$, this means $\mathcal{M} \models \varphi\ [v]$ iff $\mathcal{M}' \models \varphi\ [v]$ for every formula φ, so also for all φ of the form $P(t_1, ..., t_n)$

$\mathcal{M} \models P(t_1, ..., t_n)\ [v] \Leftrightarrow \langle [t_1]^v_{\mathcal{M}}, ..., [t_n]^v_{\mathcal{M}} \rangle \in I(P)$
$\mathcal{M}' \models P(t_1, ..., t_n)\ [v] \Leftrightarrow \langle [t_1]^v_{\mathcal{M}'}, ..., [t_n]^v_{\mathcal{M}'} \rangle \in I'(P)$

So also:

$\langle [t_1]^v_{\mathcal{M}}, ..., [t_n]^v_{\mathcal{M}} \rangle \in I(P) \Leftrightarrow \langle [t_1]^v_{\mathcal{M}'}, ..., [t_n]^v_{\mathcal{M}'} \rangle \in I'(P)$

Since this holds for any series of terms $t_1 ... t_n$, it holds when t_i is a free variable, which implies that it holds for any $d \in D$, which is condition iii) of the definition of submodel. It also holds when t_i is any individual constant, so it must also be the case that $I(a) = I'(a)$ for any individual constant a.

(ii) Counterexample:
$\mathcal{M}' = \langle D', I' \rangle$, with the single two-place predicate R.
$D' = \{1, 2, 3, 4\}$
$I'(R) = \{\langle 1, 2\rangle, \langle 1, 3\rangle, \langle 4, 2\rangle, \langle 4, 3\rangle\}$

Now we construct a submodel of $\mathcal{M}'$:

$\mathcal{M} = \langle D, I \rangle$
$D = \{1, 2\}$
$I(R) = \{\langle 1, 2\rangle\}$

Condition i) is satisfied because {1,2} is a subset of {1,2,3,4}. Condition ii) is satisfied because we have no individual constants. Condition iii) is satisfied because $\langle 1, 2\rangle \in I'(R)$, and that is the only tuple in $I(R)$ with only elements from D.

But it is not an elementary submodel, because there is a sentence that holds for $\mathcal{M}$ but not for $\mathcal{M}'$:

$\exists x \exists y(Rxy \wedge \neg \exists z(Rxz \wedge y \neq z))$

In $\mathcal{M}$ this sentence is true. When x is gets the value 1, R only holds for $\langle 1, 2\rangle$. In $\mathcal{M}'$ it is false, because R holds for $\langle 1, 2\rangle$ and $\langle 1, 3\rangle$.

(iii) $\Rightarrow$
Suppose $\mathcal{M}$ is an elementary submodel of $\mathcal{M}'$, this means $\mathcal{M} \models \varphi\ [v]$ iff $\mathcal{M}' \models \varphi[v]$

Since this holds for all formulas φ, it also holds for formulas of the form:

$\exists x \psi$

Now if $\mathcal{M}' \models \exists x\psi\ [v]$, then also $\mathcal{M} \models \exists x\psi\ [v]$. By applying the semantic definition of the existential quantifier, we know that $\mathcal{M} \models \psi\ [v(x/d)]$ for some $d \in D$. But D is a subset of D', so also $d \in D'$. Hence also $\mathcal{M}' \models \psi\ [v(x/d)]$. QED.

$\Leftarrow$
Suppose condition b) holds: For any formula $\exists x\varphi$ such that $\mathcal{M}' \models \exists x\varphi\ [v]$, there is some $d \in D$ such that $\mathcal{M}' \models \varphi\ [v(x/d)]$

If this holds, then we can prove the following by induction on the complexity of φ:

Claim: for any formula φ , $\mathcal{M} \models \varphi\ [v]$ iff $\mathcal{M}' \models \varphi[v]$, for an arbitrary [v]

Base case: Since $\mathcal{M}$ is a submodel of $\mathcal{M}'$, every atomic formula holds in $\mathcal{M}$ iff it holds in $\mathcal{M}'$, by condition a) iii) of the definition of submodel.

Induction step:
Induction hypothesis: χ and θ are equivalent in $\mathcal{M}$ and $\mathcal{M}'$.

- $\varphi = \neg\chi$
  By the induction hypothesis $\mathcal{M} \models \chi\ [v]$ iff $\mathcal{M}' \models \chi\ [v]$
  Taking the converse of this: $\mathcal{M} \not\models \chi\ [v]$ iff $\mathcal{M}' \not\models \chi\ [v]$
  This is the same as the definition of negation:
  $\mathcal{M} \models \neg\chi\ [v]$ iff $\mathcal{M}' \models \neg\chi\ [v]$
- $\varphi = \chi \vee \theta$
  By the induction hypothesis
  $\mathcal{M} \models \chi\ [v]$ iff $\mathcal{M}' \models \chi\ [v]$ and $\mathcal{M} \models \theta\ [v]$ iff $\mathcal{M}' \models \theta\ [v]$
  Then by the semantic definition of disjunction also
  $\mathcal{M} \models \chi \vee \theta\ [v]$ iff $\mathcal{M}' \models \chi \vee \theta\ [v]$
- $\varphi = \exists x\chi$
  By the induction hypothesis $\mathcal{M} \models \chi\ [v]$ iff $\mathcal{M}' \models \chi\ [v]$
  Suppose that $\mathcal{M} \models \exists x\chi\ [v]$. Let $d \in D$ be such that $\mathcal{M} \models \chi\ [v(x/d)]$.
  By the induction hypothesis, $\mathcal{M}' \models \chi\ [v(x/d)]$ (because v is arbitrary, and D is a subset of D' so $d \in D'$), and thus also $\mathcal{M}' \models \exists x\chi\ [v]$.
  On the other hand, suppose that $\mathcal{M}' \models \exists x\chi\ [v]$. Then by condition b) it holds that there is a $d \in D$ such that $\mathcal{M}' \models \chi\ [v(x/d)]$. By the induction hypothesis it follows that $\mathcal{M} \models \chi\ [v(x/d)]$. Then by the semantic definition it follows that $\mathcal{M} \models \exists x\chi\ [v]$.

The cases of conjunction, implication and the universal quantifier are left out, because formulas containing them can be rewritten to follow these cases.

(iv) By adding a fourth condition to the definition of submodel:

> iv) for all n-ary functions f:
> $I(f)(t_1, ..., t_n) = I'(f)(t_1, ..., t_n)$, if $[t_i]^v_{\mathcal{M}} \in D$ for $1 \leq i \leq n$, for any v assigning the variables onto D.

In words: for every function f in $\mathcal{M}'$ there is a function in the submodel $\mathcal{M}$ that is the restriction of f to the domain D.

Exercise 4

(i) We extend the definition of an assignment to include predicate variables:

For each predicate variable in $\mathcal{L}$, v assigns a finite subset of D.

$v(P/Q)$ is defined as follows:

$$v(X/Y)(Z) = \begin{cases} v(Z) & \text{if } Z \neq X \\ v(Y) & \text{if } Z = X \end{cases}$$

We retain all the first-order definitions, and add the following cases dealing with predicate variables:

- $\mathcal{M} \models Xt[v]$ iff $[t]^v_{\mathcal{M}} \in v(X)$
- $\mathcal{M} \models \forall X \varphi\ [v]$ iff for every finite subset Y of D, we have $\mathcal{M} \models \varphi\ [v(X/Y)]$
- $\mathcal{M} \models \exists X \varphi\ [v]$ iff for some finite subset Y of D, we have $\mathcal{M} \models \varphi\ [v(X/Y)]$

(ii) The following formula characterizes an infinite domain: [1]

$\neg \exists G \forall x G x$

There is no finite subset which contains all elements of the domain.

(iii) We can call the previous sentence φ_∞, and construct sentences φ_n meaning there are at least n objects, in the following way:
$\varphi_2 = \exists x_1 \exists x_2 : x_1 \neq x_2$
$\varphi_3 = \exists x_1 \exists x_2 \exists x_3 : x_1 \neq x_2 \wedge x_1 \neq x_3 \wedge x_2 \neq x_3$
...

Now we construct an infinite set A:

$A = \{\neg \varphi_\infty,\ \varphi_1,\ \varphi_2,\ \varphi_3,\ ...\}$

[1] Another sentence I thought of is:
$\forall F \exists G [\forall x (Fx \rightarrow Gx) \wedge \exists x (\neg Fx \wedge Gx)]$
This would show that there is always a greater set, similarly to how one proves that there are infinitely many natural numbers. But since the subsets are always finite I am not sure this sentence could characterize an infinite domain.

From this set we can take an arbitrary finite subset, and it will have a model. To see this observe that there will always be a highest n such that φ_n is in the subset, which allows the construction of a model with n elements, satisfying all sentences φ_i for $i \leq n$. Because n will be a finite number, the model will not be infinite, and $\neg\varphi_\infty$ will be satisfied as well.

If compactness would hold, the previous observation would entail that the whole set has a model. However, this cannot be the case, since the set will contain a φ-sentence for every natural number, so it is infinite, which contradicts the first sentence specifying that the model is not infinite. Hence, we have a contradiction, and we must reject the assumption that compactness would hold for second-order logic.

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