
BASIC LOGIC, HOMEWORK 4

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Exercise 1 (Prenex Normal Form)

Let \mathcal{L} be a first-order language and φ a formula of \mathcal{L} . There exists a formula ψ such that ψ is logically equivalent to φ and $\psi = Q_1 \dots Q_n \chi$, where each $Q_i \in \{\forall, \exists\}$ and χ is a formula in which no quantifiers occur.

Sketch of the proof. The theorem is proved by induction on the complexity of φ .

Base case. $\varphi = Pt_1 \dots t_n$. Take $\psi = \varphi$.

Inductive step

$\varphi = \neg\theta$: Given the induction hypothesis there is a prenex formula ρ that is logically equivalent to θ . Consider $\neg\rho$. In view of the fact that $\neg\exists x\alpha$ is logically equivalent to $\forall x\neg\alpha$, and $\neg\forall x\alpha$ to $\exists x\neg\alpha$ we can ‘push’ the negation operator in $\neg\rho$ ‘inwards, to the right’, thereby replacing \forall by \exists and vice versa. The result is a prenex formula that is logically equivalent to $\neg\theta$.

$\varphi = \theta \rightarrow \rho$: Given the induction hypothesis there are prenex formulas ζ and η that are logically equivalent to θ and ρ respectively. Consider alphabetic variants ζ' and η' of ζ and η such that no variable that occurs bound in ζ' occurs bound in η' . The formula φ is logically equivalent to $\zeta' \rightarrow \eta'$. Consider also the following fact: if x does not occur freely in α , then

$(\alpha \rightarrow \forall x\beta)$ is logically equivalent to $\forall x(\alpha \rightarrow \beta)$

$(\exists x\beta \rightarrow \alpha)$ is logically equivalent to $\forall x(\beta \rightarrow \alpha)$

$(\alpha \rightarrow \exists x\beta)$ is logically equivalent to $\exists x(\alpha \rightarrow \beta)$

$(\forall x\beta \rightarrow \alpha)$ is logically equivalent to $\exists x(\beta \rightarrow \alpha)$

In view of these facts all quantifiers in $\zeta' \rightarrow \eta'$ can be ‘pushed outwards, to the left’. The result is a prenex formula that is logically equivalent to $\zeta' \rightarrow \eta'$, and therefore to φ .

$\varphi = \exists x\theta$: Given the induction hypothesis there is a prenex formula ρ that is logically equivalent to θ . But then $\exists x\theta$ is logically equivalent to $\exists x\rho$, which is a prenex formula.

Now, the above is very sketchy. Apart from that implicit use made of a theorem that has never been explicitly stated, let alone that it has been proved. It’s up to you to repair all these defects.

Exercise 2

Let \mathcal{L} be a first-order language without identity whose only non-logical constants are a finite number of 1-place predicates. Let Δ be a set of sentences of this language.

(i) Prove the following theorem: If Δ is consistent, then Δ has a finite model.

(Hint: Given the Completeness theorem Δ has a model. Now, consider an arbitrary model $\mathcal{M} = \langle D, I \rangle$ of Δ , and transform this into a finite model of Δ by “identifying” all elements d and d' in D which have the property that for every P , $d \in I(P)$ iff $d' \in I(P)$).

- (ii) Use the above in an argument to show that the set of logical validities of \mathcal{L} is decidable.

Exercise 3 Let \mathcal{L} be a first-order language.

- (i) Prove: If φ is a sentence of \mathcal{L} such that all infinite structures are a model of φ , then there exists a finite number n such all structures with n or more elements are a model of φ .
- (ii) Disprove: If Σ is a set of sentences of \mathcal{L} such that all infinite structures are a model of Σ , then there exists a finite number n such all structures with n or more elements are models of Σ .