# Basic Logic, Homework 3

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**Exercise 1**
Let $\mathcal{L}$ be a first-order language, whose only predicate is the 2-place predicate $R$. Consider the three models $\mathcal{M}_i = \langle D_i, I_i \rangle$ $(1 \leq i \leq 3)$ defined by

$D_1 = \mathbb{N}$, $\langle d, e \rangle \in I_1(R)$ iff $d < e$;
$D_2 = \mathbb{Q}$, $\langle d, e \rangle \in I_2(R)$ iff $d < e$;
$D_3 = \mathbb{R}$, $\langle d, e \rangle \in I_3(R)$ iff $d < e$.

For each one of these models, find a sentence true in it, but false in the other two.

**Exercise 2**
In this exercise we are concerned with a first-order language $\mathcal{L}$ without function symbols, with negation $\neg$, conjunction $\wedge$, disjunction $\vee$ and the quantifiers $\forall$ and $\exists$ as its only logical constants.

(i) Let $\varphi$ be a formula of $\mathcal{L}$. Give a precise inductive definition of $\varphi^* =$ the result of interchanging conjunctions and disjunctions and likewise for the two quantifiers.

(ii) Let $\mathcal{M} = \langle D, I \rangle$ be a model for $\mathcal{L}$, and let $\mathcal{M}^*$ be the model $\langle D, I^* \rangle$, which is just like $\mathcal{M}$, except that for any $n$-ary predicate $P$, $\langle d_1 \dots, d_n \rangle \in I^*(P)$ iff $\langle d_1, \dots, d_n \rangle \notin I(P)$. Show that for any sentence $\varphi$,

$$\mathcal{M} \models \varphi \text{ iff } \mathcal{M}^* \models \neg(\varphi^*)$$

**Exercise 3**
Let $\mathcal{L}$ be a language of first-order logic with equality whose only non-logical symbol is one 1-place predicate $P$. Consider the set of sentences $\Sigma$ given by:

$$\Sigma = \{\exists x Px, \exists x \neg Px, \forall x \forall y \forall z (x = y \vee x = z \vee y = z)\}$$

Show that $\Sigma$ is consistent and complete.

(Reminder: By definition, $\Sigma$ is *complete* iff for every sentence $\varphi$ either $\Sigma \vdash \varphi$ or $\Sigma \vdash \neg\varphi$)

**Exercise 4** Let $\mathcal{L}$ be a first-order language without function symbols, and $P$ an $n$-place predicate. An *explicit definition* of $P$ is a sentence of the form

$$\forall x_1 \dots \forall x_n (P(x_1, \dots, x_n) \leftrightarrow \varphi)$$

Here, $\varphi$ can be any formula in which $P$ does not occur, and in which at most the variables $x_1, \dots, x_n$ occur free.

Now, let $\Sigma$ be a theory of $\mathcal{L}$. By definition, a predicate $P$ is *eliminable* from $\Sigma$ iff there exists an explicit definition $\epsilon$ of $P$ such that $\Sigma \models \epsilon$.

(i) Assume that $P$ is eliminable from $\Sigma$ by the explicit definition $\epsilon$.
Consider $\Sigma'$ given by

$$\Sigma' = \{\varphi \mid P \text{ does not occur in } \varphi, \text{ and } \Sigma \models \varphi\} \cup \{\epsilon\}$$

Show that for all $\psi$, $\Sigma \models \psi$ iff $\Sigma' \models \psi$.
(Hint: Prove first that for every formula $\chi$ in which $P$ occurs there exists a formula $\chi'$ in which $P$ does not occur such that $\Sigma' \models \chi \leftrightarrow \chi'$)

(ii) Here is an example. Let $\Sigma$ consist of the following sentences:
$\forall x \forall y \forall z((Rxy \wedge Ryz) \to Rxz)$
$\forall x \forall y \forall z((Exy \wedge Eyz) \to Exz)$
$\forall x \neg Rxx$
$\forall x \forall y((Rxy \vee Ryx) \to \neg Exy)$
$\forall x \forall y((Rxy \vee Ryx) \vee Exy)$

(Read '$Rxy$' as 'x is prefered to y', and '$Exy$' as 'x and y are equally prefered' )

(a) Show that $E$ is eliminable from $\Sigma$.
(b) Show that $R$ is not eliminable from $\Sigma$.