Basic Logic 5

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2 Semantics

Definition 5 (Model) *Let* $\mathcal{L}$ *be a language of first order predicate logic. A* model $\mathcal{M}$ *for* $\mathcal{L}$ *is a pair* $\langle D, I\rangle$ *with the following properties.*

(i) $D \neq \emptyset$

(ii) I *is a function assigning*

(a) an element of D *to each individual constant of* $\mathcal{L}$*;*

(b) a subset of D^n *to each* n*-ary predicate of* $\mathcal{L}$*;*

(c) a function from D^n *into* D *to each* n*-ary function symbol of* $\mathcal{L}$*.*

The set D is called the *domain (of discussion)* of $\mathcal{M}$ and I specifies the *interpretation* of the non-logical constants of $\mathcal{L}$ in $\mathcal{M}$. Clause *(ii)(a)* says that that the interpretation $I(a)$ of an individual constent a is an element D, which means so much as that the individual constants function as *names* for the objects in D.[1] A one-place predicate P is interpreted as a subset of D: de set of objects possessing the property expressed by P. The interpreatation of a two-place predicate R is a subset of $D \times D$, a set of ordered pairs — intuitively the set of all pairs of objects between which the relation expressed by R exists.

Given a language $\mathcal{L}$ and a model $\mathcal{M}$ the truth value of the sentences of $\mathcal{L}$ are fixed. We now want to specify their truth values, and we want to do so in a compositional way.

One of the obstacles here is that sentences are often built from formulas with free variables. For example, if want to spell out the truth condition of a sentence

[1] Notice that we do not require that for every element $d \in D$ there is some individual constant a such that $I(a) = d$: objects can be nameless.

of the form $\forall x\varphi$ in a compositional way we will have to do so in terms of the 'value' of φ, but intuitively, if x occurs free in φ, its value, whatever it may be, is not a truth value.

Suppose somebody were to ask you what the truth value is of the phrase

$$n \leq 5$$

Probably you would think this is a silly question. The phrase '$n \leq 5$' does not have a truth value, its truth value *depends* of the value assigned to n and this value is not fixed, it varies.

Well, then, this is exactly the way we will treat the formulas with free variables of our formal languages. Such a formula can be true one time and false the other depending on the values assigned to the free variables occurring in the formula. This way we can spell out the truth condition of the sentence $\forall x\varphi$ as follows" The sentence $\forall x\varphi$ is true in $\mathcal{M}$ iff for every $d \in D$ it holds that if d is the object assigned to x the formula φ gets true in $\mathcal{M}$.

From here to the truth definition is still a long way. But the formalities that follow are really just a way to precisify this idea.

Definition 6 (Assignment) *Let $\mathcal{L}$ be a language of first order predicate logic and $\mathcal{M} = \langle D, I\rangle$ a model for $\mathcal{L}$. An* assignment *v is a function that assigns an element of D to each variable of $\mathcal{L}$. is a pair with the following properties.*

If v is an assignment, x a variable, and d an element of D, then $v(x/d)$ is the assignment v' defined by (i) if $y \neq x$ then $v'(y) = v(y)$, and (ii) $v'(x) = d$.

These assignments — of values to the free variables in φ — will be the things that the truth value of φ depend on.

Definition 7 (The semantics of terms) *Let $\mathcal{L}$ be a language of first order predicate logic and $\mathcal{M} = \langle D, I\rangle$ a model for $\mathcal{L}$. We define for every term t of $\mathcal{L}$,* the value $[t]^v_{\mathcal{M}}$ of t in $\mathcal{M}$ under v *as follows.*

(i) if t is an individual constant of $\mathcal{L}$, then $[t]^v_{\mathcal{M}} = I(t)$;

(ii) if t is an individual constant of $\mathcal{L}$, then $[t]^v_{\mathcal{M}} = v(t)$;

(iii) if $t = f(t_1, \ldots, t_n)$ with f an n-ary function symbol and $t_1, \ldots, t_n$ terms, then $[t]^v_{\mathcal{M}} = I(f)([t_1]^v_{\mathcal{M}}, \ldots, [t_n]^v_{\mathcal{M}})$.

Example Let '+' be a two-place function symbol of $\mathcal{L}$. We will write '$(t + t')$' rather than '$+(t, t')$'. Let a be an individual constant of $\mathcal{L}$. Consider the model $\mathcal{M} = \langle D, I\rangle$ given by $D = \mathbb{N}$, $I(+) =$ the addition operator on the natural numbers, $I(a) = 0$. Consider an assignment v such that $v(x) = v(y) = 1$ and $v(z) = 23$.

Compute $[x + a]^v_{\mathcal{M}}$, $[y + z]^v_{\mathcal{M}}$, $[((x + y) + (z + a))]^v_{\mathcal{M}}$.

Definition 8 (The semantics of formulas) *Let $\mathcal{L}$ be a language of first order predicate logic and $\mathcal{M} = \langle D, I \rangle$ a model for $\mathcal{L}$. We define for every formula tφ of $\mathcal{L}$, what it means for φ to be* true in $\mathcal{M}$ under v, *writing*

$$\mathcal{M} \models \varphi \ [v]$$

when this relation holds.

(i) If $\varphi = Q(t_1, \ldots, t_n)$, then $\mathcal{M} \models \varphi \ [v]$ iff $\langle [t_1]^v_{\mathcal{M}}, \ldots, [t_n]^v_{\mathcal{M}} \rangle \in I(Q)$, with as a special case:
If $\varphi = (t_1 = t_2)$, then $\mathcal{M} \models \varphi \ [v]$ iff $[t_1]^v_{\mathcal{M}} = [t_2]^v_{\mathcal{M}}$.[2]

(ii) If $\varphi = \neg\psi$, then $\mathcal{M} \models \varphi \ [v]$ iff $\mathcal{M} \not\models \psi \ [v]$.

(iii) If $\varphi = \psi \wedge \chi$, then $\mathcal{M} \models \varphi \ [v]$ iff $\mathcal{M} \models \psi \ [v]$ and $\mathcal{M} \models \psi \ [v]$.

(iv) If $\varphi = \psi \vee \chi$, then $\mathcal{M} \models \varphi \ [v]$ iff $\mathcal{M} \models \psi \ [v]$ or $\mathcal{M} \models \psi \ [v]$.

(v) If $\varphi = \psi \rightarrow \chi$, then $\mathcal{M} \not\models \varphi \ [v]$ iff $\mathcal{M} \models \psi \ [v]$ or $\mathcal{M} \models \psi \ [v]$.

(vi) If $\varphi = \forall x\psi$, then $\mathcal{M} \models \varphi \ [v]$ iff $\mathcal{M} \models \psi \ [v(x/d)]$ for every $d \in D$.

(vii) If $\varphi = \exists x\psi$, then $\mathcal{M} \models \varphi \ [v]$ iff $\mathcal{M} \models \psi \ [v(x/d)]$ for some $d \in D$.

Example
Let R be a two-place predicate and a be an individual constant of $\mathcal{L}$. Consider the model $\mathcal{M} = \langle D, I \rangle$ given by $D = \{1, 2\}$, $I(R) = \{\langle 1, 2 \rangle, \langle 2, 2 \rangle, \langle 2, 1 \rangle\}$ and $I(a) = 1$. Consider an assignment v such that $v(x) = 1$ and $v(y) = 2$.

Determine whether $\mathcal{M} \models R(a, a) \ [v]$. Does the assignment v matter to the outcome?
$\mathcal{M} \models R(x, x) \ [v]$. Does the assignment v matter to the outcome?
$\mathcal{M} \models \forall z R(z, a) \ [v]$. Does the assignment v matter to the outcome?
$\mathcal{M} \models \forall z R(z, y) \ [v]$. Does the assignment v matter to the outcome?
$\mathcal{M} \models \exists x \forall y R(x, y) \ [v]$. Does the assignment v matter to the outcome?

In the next proposition it is shown that the truth value of a formula under a given assignment v depends only on the values assigned to the free variables occurring in that formula — that's how we meant things to come out, so we are just checking that they do.

Proposition 2 *Let $\mathcal{L}$ be a language of first order predicate logic and $\mathcal{M} = \langle D, I \rangle$ a model for $\mathcal{L}$. Let α be a term or formula in which at most the variables $x_1, \ldots, x_n$ occur freely.*

Consider two assignments v and v' such that for all $1 \leq i \leq n$ $v(x_i) = v'(x_i)$. Then the following holds:

[2] This is confusing because I am using the identity sign '=' both in the object language and in the metalanguage.

(i) If α is a term, then $[\alpha]^v_{\mathcal{M}} = [\alpha]^{v'}_{\mathcal{M}}$

(ii) If α is a formula, then $\mathcal{M} \models \alpha\ [v]$ iff $\mathcal{M} \models \alpha\ [v']$

Proof.
(i) The proof is by induction on the complexity of α, and left to the reader.
(ii) This, too is proved by induction on the complexity of α.

Base case: Let $\alpha = Q(t_1, \dots, t_n)$. Then we have

$$\mathcal{M} \models \alpha\ [v] \text{ iff } \langle [t_1]^v_{\mathcal{M}}, \dots, [t_n]^v_{\mathcal{M}} \rangle \in I(Q)$$

By (i) we know that

$$[t_i]^v_{\mathcal{M}} = [t_i]^{v'}_{\mathcal{M}} \text{ for all } 1 \leq i \leq n$$

So,

$$\mathcal{M} \models \alpha\ [v] \text{ iff } \langle [t_1]^{v'}_{\mathcal{M}}, \dots, [t_n]^{v'}_{\mathcal{M}} \rangle \in I(Q),$$

which means that

$$\mathcal{M} \models \alpha\ [v] \text{ iff } \mathcal{M} \models \alpha\ [v']$$

*Induction step:*The only interesting cases are $\alpha = \forall x \varphi$ and $\alpha = \exists x \varphi$. We leave the first to the reader. Consider the case that $\alpha = \exists x \varphi$. Suppose that $\mathcal{M} \models \alpha\ [v]$. This is so iff there is some $d \in D$ such that $\mathcal{M} \models \varphi\ [v(x/d)]$. The assignment v' coincides v coincides with v as far as the free variables in $\exists x \varphi$ are concerned. Given this, it follows that the assignment $v'(x/d)$ coincides with $v(x/d)$ as far as the free variables in φ are concerned. Hence by the induction hypothesis we find that $\mathcal{M} \models \varphi\ [v'(x/d)]$, which means that $\mathcal{M} \models \alpha\ [v']$.

Corollary 1 *Let φ be a sentence. Then*

$$\mathcal{M} \models \varphi\ [v] \textit{ for some } v \textit{ iff } \mathcal{M} \models \alpha\ [v] \textit{ for all } v$$

In the sequel we will write $\mathcal{M} \models \varphi$ — and say 'φ is true in $\mathcal{M}$' — when $\mathcal{M} \models \varphi\ [v]$ for all v.

We now have to deal with a number of rather tedious propositions, all pertaining to the substitution of terms for other terms. However, we cannot skip them on our way to more interesting results. In these propositions we compare the truth conditions of a formula φ with the truth conditions of the formula $[t/x]\varphi$.

Proposition 3
Let $\mathcal{L}$ be a language of first order predicate logic and $\mathcal{M} = \langle D, I \rangle$ a model for $\mathcal{L}$. Let t be a term which is free for x inφ. Then

$$\mathcal{M} \models [t/x]\varphi\ [v] \textit{ iff } \mathcal{M} \models \varphi\ [v(x/[t]^v_{\mathcal{M}})]$$

Exercise 9 *Show that the above proposition does not hold if the requirement that t is free for x in φ is omitted.*

Exercise 10 *Prove the above proposition. It is one of those propositions the truth of which is immediately clear, whereas the proof is still a lot work. Proceed by distinguishing the following cases:*

(i) *x does not occur freely in φ.*

(ii) *x occurs free in φ. In this case you have to use induction on the complexity of φ. For the case that $\varphi = \exists y\psi$ you have to distinguish two cases (a) y does not occur free in ψ, and (ii) y occurs free in ψ.*

Proposition 4
Let $\mathcal{L}$ be a language of first order predicate logic and $\mathcal{M} = \langle D, I\rangle$ a model for $\mathcal{L}$. Let y be a variable which is free for x inφ, and which does not occur free in ψ. Then

$$\mathcal{M} \models \varphi\ [v] \textit{ iff } \mathcal{M} \models [y/x]\varphi\ [v(y/v(x)]$$

Proof.
Due to the fact that y does not occur free in φ, $[x/y][y/x]\varphi = \varphi$. Now, set $\psi = [y/x]\varphi$ then we can write $[x/y]\psi$ for φ. Applying proposition 3 we find

$$\mathcal{M} \models [x/y]\psi\ [v] \text{ iff } \mathcal{M} \models \psi\ [v(y/v(x)]$$

But by the above definitions this is equivalent to

$$\mathcal{M} \models \varphi\ [v] \text{ iff } \mathcal{M} \models [y/x]\varphi\ [v(y/v(x)]$$

Exercise 11 *Show that the above proposition does not hold if the requirement that y does not occur free in φ is omitted.*

Exercise 12 *Suppose that the variables $x_1, \ldots, x_n$ occur freely in φ. The* universal closure *$\overline{\varphi}$ of φ is defined as the sentence $\forall x_1, \ldots, \forall x_n\varphi$. Prove that*

$$\mathcal{M} \models \varphi \textit{ iff } \mathcal{M} \models \overline{\varphi}$$

Definition 9 *(Alphabetic variant)*
We define inductively when a formula ψ is an alphabetic variant of a formula φ.

- *if φ is atomic, then ψ is an alphabetic variant of φ if and only if $\psi = \varphi$.*

- *if $\varphi = \neg\chi$ then ψ is an alphabetic variant of φ iff $\psi = \neg\chi'$ with χ' an alphabetic variant of χ.*

- *if $\varphi = \chi \wedge \theta/\chi \vee \theta/\chi \to \theta$, then ψ is an alphabetic variant of φ iff $\psi = \chi' \wedge \theta'/\chi' \vee \theta'/\chi' \to \theta'$ with χ' and θ' alphabetic variants of χ and θ.*

- *if $\varphi = \forall x\chi/\exists x\chi$ then ψ is an alphabetic variant of φ iff $\psi = \forall y[y/x]\chi'/\exists y[y/x]\chi'$ where y is free for x in χ', y does not occur freely in χ', and χ' is an alphabetic variant of χ*

Exercise 13 *Prove the following.*
If ψ is an alphabetic variant of φ, then $\mathcal{M} \models \varphi\ [v]$ iff $\mathcal{M} \models \psi\ [v]$.

Definition 10 *(Logical notions)*
Let φ, ψ be formulas and Δ a set of formulas.

(i) *φ is* logically equivalent *to ψ iff for all models $\mathcal{M}$, and for all assignmentsv, $\mathcal{M} \models \varphi$ iff $\mathcal{M} \models \psi$*

(ii) *φ* follows from *Δ iff for all models $\mathcal{M}$, and for all assignments v, if $\mathcal{M} \models \psi\ [v]$ for all $\psi \in \Delta$, then $\mathcal{M} \models \varphi\ [v]$.*

Instead of '*φ follows from Δ*' we will often write '$\Delta \models \varphi$', and say '*the argument Δ/φ is logically valid*'.

Warning
In the literature you may find a different definition of logical validity, as follows:

$\Delta \models^* \varphi$ iff for all models $\mathcal{M}$ if $\mathcal{M} \models \psi$ for all $\psi \in \Delta$, then $\mathcal{M} \models \varphi$.

Notice

(i) If $\Delta \models \varphi$, then $\Delta \models^* \varphi$.

(ii) The converse does not hold $Px \models^* \forall xPx$, but $Px \not\models \forall xPx$.

(iii) If $\Delta, \varphi \models \psi$, then $\Delta \models \varphi \rightarrow \psi$

(iv) it is not necessarily so that If $\Delta, \varphi \models^* \psi$, then $\Delta \models^* \varphi \rightarrow \psi$

(v) If φ and all $\psi \in \Delta$ are sentences, then If $\Delta \models^* \varphi$, then $\Delta \models \varphi$

So, for sentences it does not matter whether one works with *validity** or *validity*

We are ready now to prove the correctness theorem for our natural deduction system.

Theorem 1 *Let φ be a formula and Δ a set of formulas of some language of first order predicate logic (with identity).*
If $\Delta \vdash \varphi$, then $\Delta \models \varphi$

Exercise 14 *Prove this theorem.*