Basic Logic 2

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2 Proofs about proofs

If things are right, you start reading this section after having been introduced to the system of natural deduction treated in Chapter 4 of the Gamut textbook[1]. So, you know how to make derivations *in* that system, and you will have made a number of exercises of the following kind:

Exercise 8 *Give natural deductions showing that*

(a) $\vdash (p \to \neg p) \to ((q \to p) \to \neg q)$

(b) $(q \vee p) \wedge (q \vee \neg p) \vdash q$

(c) $\vdash (p \to q) \vee (\neg p \to \neg q)$

(d) $p \to (q \to r), \neg(p \to s), \neg q \to s, (r \wedge q) \to s \vdash \bot$

Maybe you already saw some theorems *about* that system, theorems like the one mentioned in the following exercise.

Exercise 9 *If there exists a natural deduction of φ from Δ, then there exists a natural deduction of φ from Δ in which the rule EFSQ is not used.*

In this section we will prove more theorems *about* the natural deduction system. Our main concern will be to compare the syntactic notion of *derivability* with the semantic notion of *validity*.
As we already did in the exercises above, we write

$$\Delta \vdash \varphi$$

[1] Gamut, L.T.F., *Language, Logic and Meaning, vol. 1: Introduction to Logic*, University of Chicago Press, 1991

if there exists a natural deduction with conclusion φ from a number of premises taken from the set Δ. We often say in such a case that φ is derivable from Δ. Don't confuse the expression '$\Delta \vdash \varphi$' with the expression

$$\Delta \models \varphi$$

Recall that the latter means that φ *is entailed by* Δ in the semantic sense of the word: for every interpretation I it holds that if $V_I(\psi) = 1$ for every $\psi \in \Delta$, then $V_I(\varphi) = 1$.

Even though the relations '$\vdash$' and '$\models$' have different meanings, they amount to the same thing. It's our main objective in this chapter to show this. In other words, we want to show that the system of natural deduction is so construed that the notion of derivabilty coincides with the notion of validity. The *correctness theorem* says that everything that is provable is valid. According to the *completeness theorem* the converse holds as well.

We will prove more theorems in this chapter than just these two. Here our choice is mainly determined by didactic considerations. It concerns theorems that are proved using the same methods that we will use proving correctness and completeness.

2.1 The correctness theorem

The correctness theorem states that if a sentence is derivable from a set of premises, it also follows semantically from this set of premises.

Theorem 1 (Correctness Theorem) $\Delta \vdash \varphi \Rightarrow \Delta \models \varphi$

Let's start by picturing exactly what a natural deduction *is*. A natural deduction consists of a finite number of *steps*. These steps are numbered. At each of these steps one of the following three things can be done.

1. Introduce a *premise*.
2. Make an *assumption*. Every assumption must be withdrawn later in the deduction. This can be done by applying one of the rules $I_\rightarrow$, or $I_\neg$. An assumption made at step k can only be dropped at step l if all assumptions made between k and l have already been dropped. If an assumption made before step k has not yet been dropped at step k, we say that the assumption is *active* at step k. The set of assumptions active at step k is denoted by Γ_k.
3. Apply one of the rules $I_\wedge$, $E_\wedge$, $I_\vee$, $E_\vee$, $I_\rightarrow$, $E_\rightarrow$, $I_\neg$, $E_\neg$, $EFSQ$, $\neg\neg$ or *rep*.

To prove correctness we have to show that the sentence φ derived at the last step of a natural deduction follows form the premises introduced at some step of the deduction. We will prove something stronger:

Proposition *Consider an arbitrary derivation. For each step n in this derivation, the sentence introduced at step n, follows logically from the premises used in the derivation* plus *the assumptions that are active at step n.*

The proposition is proved by induction on n. This proof method, called induction on the length of a derivation, will be used many times in this course, and therefore it is really important that you understand its ins and outs.

Each and every time this method is used the goal is to show that a certain proposition holds for all steps n in an arbitrary derivation. And each and every time the goal is achieved by showing two things:

Base case We show that the proposition at issue holds for step 1.

Inductive step We show that for al steps n the following holds: *If* the proposition at issue holds for all steps $k \leq n$, *then*, it holds for step $n+1$, too.

If one succeeds proving both the base case and the inductive step, one may conclude that the proposition holds for all steps. (Given the base case the proposition holds for $n = 1$; by the inductive step it follows that the proposition also holds for $n = 2$; applying the inductive step once more, we learn that the proposition holds for $n = 3$, etc.)

In this specific case the proof looks as follows:

Base case: At the first step in a derivation one can only do one of two things

(a) introduce a premise;

(b) introduce an assumption.

Let φ_1 be the sentence entered. In both cases it follows immediately that $\Delta, \Gamma_1 \models \varphi_1$, because $\varphi_1 \in \Delta \cup \Gamma_1$.

Inductive step: Assume the proposition holds for all steps $k \leq n$ (Induction hypothesis). Consider step $n+1$. The following possibilities occur:

(a) At step $n+1$ a premise is introduced.

(b) At step $n+1$ an assumption is made.

Just like above, in these two cases things follow immediately.

(c) At step $n+1$ the sentence $\bot$ is introduced by an application of $E_\neg$. In this case there are $j \leq n$ and $k \leq n$ such that

(i) At step j some sentence ψ and at step k the sentence $\neg\psi$ is introduced, while

(ii) all assumptions active at j and k are also active at step $n+1$. Otherwise it would not have been allowed to apply $E_\neg$.

So, we find that $\Gamma_j \subseteq \Gamma_{n+1}$ and $\Gamma_k \subseteq \Gamma_{n+1}$. Now, let Δ be the set of premises used in the derivation. De induction hypothesis tells that $\Delta \cup \Gamma_j \models \psi$ and $\Delta \cup \Gamma_k \models \neg\psi$. Given the fact that $\Gamma_j \cup \Gamma_k \subseteq \Gamma_{n+1}$, this means that there is no I such that $V_I(\chi) = 1$ for all $\chi \in \Delta \cup \Gamma_{n+1}$. Hence, we have that $\Delta \cup \Gamma_{n+1} \models \bot$.

(d) At step $n+1$ a sentence ψ is introduced by an application of $EFSQ$. In this case there will be some $k \leq n$ such that

(i) At step k the sentence $\bot$ is introduced.

(ii) All assumptions active at step k are also active at step $n+1$. Otherwise, it would not have been allowed to apply $EFSQ$.

So, $\Gamma_k \subseteq \Gamma_{n+1}$. Now let Δ be the set of premises used in the derivation. The induction hypothesis tells us that $\Delta \cup \Gamma_k \models \bot$. This can only be true if there is no I such that $V_I(\chi) = 1$ for all $\Delta \cup \Gamma_k$. Given the fact that $\Gamma_k \subseteq \Gamma_{n+1}$, it follows that there is no I such that $V_I(\chi) = 1$ for all $\Delta \cup \Gamma_{n+1}$, which means that $\Delta \cup \Gamma_{n+1} \models \psi$ for any ψ.

(e) At step $n + 1$ a sentence of the form $\neg\psi$ is introduced by an application of $I_\neg$. In this case we have the following:

(i) $\Gamma_n = \Gamma_{n+1} \cup \{\psi\}$, because if $\neg\psi$ is introduced by an application of $I_\neg$, then the assumption ψ is dropped and no new assumption is made.

(ii) At step n the sentence $\bot$ has been derived. The induction hypothesis tells us that $\Delta \cup \Gamma_n \models \bot$; In other words, $\Delta \cup \Gamma_{n+1} \cup \{\psi\} \models \bot$. It is not difficult to see that this means that $\Delta \cup \Gamma_{n+1} \models \neg\psi$.

(f) At step $n + 1$ $I_\wedge$ is applied. Left to the reader.

(g) At step $n + 1$ $E_\wedge$ is applied. Left to the reader.

(h) At step $n + 1$ $I_\vee$ is applied. Left to the reader.

(i) At step $n+1$ a sentence ψ is introduced by an application of $E_\vee$. In this case there are steps $i \leq n$, $j \leq n and k \leq n$ such that:

(i) At step i a sentence of the form $(\chi \vee \theta)$ has been introduced, at step j the sentence $(\chi \to \psi)$, and at step k the sentence $(\theta \to \psi)$, while

(ii) $\Gamma_i \cup \Gamma_j \cup \Gamma_k \subseteq \Gamma_{n+1}$. Given the induction hypothesis we may assume that $\Delta \cup \Gamma_i \models (\chi \vee \theta)$, $\Delta \cup \Gamma_j \models (\chi \to \psi)$ and $\Delta \cup \Gamma_k \models (\theta \to \psi)$. From this it follows without much ado that $\Delta \cup \Gamma_{n+1} \models \psi$

Exercise 10 *There are a number of paragraphs in the above that start with words like "It is not difficult to see that..." Maybe you did have difficulties seeing that. That's why we list these observations here together with some other. Taken together they make up a lemma, that we could (or even should) have proved before the correctness theorem.*

(a) *If* $\Delta \models \psi$ *and* $\Delta \models \neg\psi$, *then* $\Delta \models \bot$ *(already proved under (c) above);*

(b) *If* $\Delta \models \bot$, *then* $\Delta \models \psi$ *(already proved under (d) above);*

(c) *If* $\Delta, \psi \models \bot$, *then* $\Delta \models \neg\psi$;

(d) *If* $\Delta \models \psi$ *and* $\Delta \models \chi$, *then* $\Delta \models \psi \wedge \chi$;

(e) *If* $\Delta \models \psi \wedge \chi$, *then* $\Delta \models \psi$ *and* $\Delta \models \chi$;

(f) *If* $\Delta \models \psi$, *then* $\Delta \models \psi \vee \chi$;

(g) *If* $\Delta \models (\chi \vee \theta)$ *and* $\Delta \models (\chi \rightarrow \psi)$ *and* $\Delta \models (\theta \rightarrow \psi)$, *then* $\Delta \models \psi$;

(h) *If* $\Delta, \psi \models \chi$, *then* $\Delta \models \psi \rightarrow \chi$;

(i) *If* $\Delta \models \psi$ *and* $\Delta \models \psi \rightarrow \chi$, *then* $\Delta \models \chi$;

(j) *If* $\Delta \models \neg\neg\psi$, *then* $\Delta \models \psi$;

Exercise 11 *Try to find out which steps are missing in the proof of the Correctness Theorem. Fill in three of them).*

Exercise 12
In this exercise we compare the classical natural deduction system for languages with $\wedge$, $\vee$, $\neg$, $\rightarrow$ *and* $\bot$ *as connectives with the intuitionistic natural deduction system for these languages. As you already know from the Gamut book, the difference between the two systems is that in the former it is allowed to apply the* $\neg\neg$*-rule and in the latter it is not.*

The theorem you have to prove is the following:

Theorem*(Gödel)*
For all Δ *and* φ, φ *is derivable from* Δ *in classical logic iff* $\neg\neg\varphi$ *is derivable from* Δ *in intuitionistic logic.*

The proof from right to left is easy. The converse is proved by induction on the number of steps in the derivation. The following lemma will be helpful.

Lemma
Below, '$\Delta \vdash_i \psi$' stands for 'there exists an intuitionistic deduction of ψ from Δ.'

(i) $\neg\neg\perp \vdash_i \perp$

(ii) $\neg\neg\varphi, \neg\neg\psi \vdash_i \neg\neg(\varphi \wedge \psi)$

(iii) $\neg\neg(\varphi \wedge \psi) \vdash_i \neg\neg\psi$
$\neg\neg(\varphi \wedge \psi) \vdash_i \neg\neg\varphi$

(iv) $\neg\neg\varphi \vdash_i \neg\neg(\varphi \vee \psi)$
$\neg\neg\psi \vdash_i \neg\neg(\varphi \vee \psi)$

(v) $\neg\neg(\varphi \vee \psi), \neg\neg(\varphi \to \chi), \neg\neg(\psi \to \chi) \vdash_i \neg\neg\chi$

(vi) $\varphi \to \neg\neg\psi \vdash_i \neg\neg(\varphi \to \psi)$

(vii) $\neg\neg\varphi, \neg\neg(\varphi \to \psi) \vdash_i \neg\neg\psi$

(viii) $\neg\neg\neg\neg\varphi \vdash_i \neg\neg\varphi$

Beware: these derivations are tricky. Here are two examples, just to give you an idea of how to go about.

Proof of (viii).

1.	$\neg\neg\neg\neg\varphi$	prem.
2.	$\neg\varphi$	ass.
3.	$\neg\neg\varphi$	ass.
4.	$\perp$	$E_\neg$ 2, 3
5.	$\neg\neg\neg\varphi$	$I_\neg$
6.	$\perp$	$E_\neg$ 1, 5
7.	$\neg\neg\varphi$	$I_\neg$

Proof of (ii).

1.	$\neg\neg\varphi$	prem.
2.	$\neg\neg\psi$	prem.
3.	$\neg(\varphi \wedge \psi)$	ass.
4.	φ	ass.
5.	ψ	ass.
6.	$\varphi \wedge \psi$	$I_\wedge$ 4, 5
7.	$\perp$	$E_\neg$ 3, 6
8.	$\neg\psi$	$I_\neg$
9.	$\perp$	$E_\neg$ 2,8
10.	$\neg\varphi$	$I_\neg$
11.	$\perp$	$E_\neg$ 1, 10
12.	$\neg\neg(\varphi \wedge \psi)$	$I_\neg$

Corollary 1 No classical logical truth counts as a 'logical contingency' in intuitionistic logic. Formally: there is no φ such that $\vdash \varphi$ and $\neg\varphi \nvdash_i \bot$.

Corollary 2 If $\Delta \vdash \varphi$, then there exists a derivation of φ from Δ in which the $\neg\neg$-rule is used at most once.

2.2 Axiomatic Systems

There are other proof methods for propositional logic in addition to natural deduction. Before the invention of natural deduction all proof systems were 'axiomatic' in nature. In this section we will study one such axiomatic system more closely.

We will start with an axiom system for just negation, implication and falsum, and later extend it to other logical constants.

Definition 5 *Let $\mathcal{L}$ be a language with $\bot$, $\neg$ en $\rightarrow$ as logical constants. A sentence θ is a* axiom *of $\mathcal{L}$ iff θ satisfies one of the following conditions:*

(i) $\theta = \varphi \rightarrow (\psi \rightarrow \varphi)$

(ii) $\theta = (\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi))$

(iii) $\theta = \neg\varphi \rightarrow (\varphi \rightarrow \bot)$

(iv) $\theta = (\varphi \rightarrow \bot) \rightarrow \neg\varphi$

(v) $\theta = \bot \rightarrow \varphi$

(vi) $\theta = \neg\neg\varphi \rightarrow \varphi$

An *axiomatic deduction* of a sentence φ from a set of premises Δ consists of a finite number of *steps* that are numbered: $1, 2, 3, \ldots, n$. At every step k a sentence ψ is *introduced*, and for every step k it holds that the sentence introduced at k must satisfy one of the following conditions:

1. $\psi \in \Delta$;

2. ψ is an axiom;

3. ψ can be derived from two sentences introduced earlier by an application of *modus ponens*, the rule that we have called $E_{\rightarrow}$ in the above.

Just like in the case of natural deduction, the sentence introduced at the last step of an axiomatic deduction is called the *conclusion*.

Example What follows is an axiomatic deduction of $(p \rightarrow p)$ from the empty set of premises:

1.	$p \to ((p \to p) \to p)$	axioma (i)
2.	$(p \to ((p \to p) \to p)) \to ((p \to (p \to p)) \to (p \to p))$	axioma (ii)
3.	$(p \to (p \to p)) \to (p \to p)$	1, 2, mp
4.	$p \to (p \to p)$	axioma (i)
5.	$p \to p$	3, 4, mp

I hope you don't need more examples to get convinced that developing an axiomatic deduction, even for the simplest arguments, can be quite a complicated affair. What makes things so complicated is the fact that no *assumptions* can be made and dropped. Still, this axiom system is just as powerful as the system of natural deduction for the logical constants $\to$, $\neg$ and $\bot$, which consists of the rules $E_\to$, $I_\to$, $E_\neg$, $I_\neg$, $EFSQ$, $\neg\neg$ en *rep*.

In the proof of this statement we will appeal to the so-called *Deduction Theorem*, which is one of the first proof theoretic theorems ever proved. In the proof of the latter we will use the notation '$\Delta \vdash_a \varphi$', as short for 'there exists an axiomatic deduction for φ from Δ'.

Theorem 2 (Deduction theorem) *If* $\Delta, \varphi \vdash_a \psi$*, then* $\Delta \vdash_a \varphi \to \psi$*.*

Proof assume that $\Delta, \varphi \vdash_a \psi$. Consider an axiomatic deduction ofψ from $\Delta \cup \{\varphi\}$. Letχ be the sentence derived at step k. We prove by induction on k that $\Delta \vdash_a \varphi \to \chi$. *A fortiori* it follows that $\Delta \vdash_a \varphi \to \psi$.

Basis: $k = 1$. We must distinguish the following possibilities:

(a) The sentence χ introduced at step 1, is an element of Δ.

(b) The sentence χ introduced at step 1, is φ.

(c) The sentence χ introduced at step 1, is an axiom.

Each of these possibilities also shows up at the induction step, and is discussed there.

Induction step: $n = k + 1$. Assume it holds for all $i \leq k$ that $\Delta \vdash_a \varphi \to \chi$ for χ the sentence introduced at step i. We have to show that $\Delta \vdash_a \varphi \to \chi$ for χ the sentence introduced at step $k + 1$. The following possibilities occur:

(a) The sentence χ introduced at step $k+1$ is an axioma. The induction hypothesis is not needed to show that $\Delta \vdash_a \varphi \to \chi$. First, we can be sure that there exists an axiomatic deduction of χ from Δ. Add to this deduction in a second step the axiom $\chi \to (\varphi \to \chi)$ and in a third step $\varphi \to \chi$, deduced from the first two by an application of *Modus Ponens*, thus obtaining an axiomatic deduction of $\varphi \to \chi$ from Δ.

(b) The sentence χ introduced at step $k + 1$ is an element of Δ. In this case one can proceed in the same way as in (a).

(c) $\chi = \varphi$. Here, too we don't need the induction hypothesis. We already know (see page 7) that $\vdash_a \varphi \to \varphi$, which implies that $\Delta \vdash_a \varphi \to \varphi$.

(d) The sentence χ introduced at step $k+1$ is obtained by an application of *Modus Ponens*. In this case there are $i, j \leq k$ such that at step i a sentence of the form $\theta \to \chi$ and at step j the sentence θ are obtained. In view of the induction hypothesis we may assume that $\Delta \vdash_a \varphi \to \theta$ and $\Delta \vdash_a \varphi \to (\theta \to \chi)$. An axiomatic deduction of $(\varphi \to \chi)$ from Δ is obtained by first giving a deduction in which both $\varphi \to \theta$ and $\varphi \to (\theta \to \chi)$ are derived from Δ. Next add to this the following steps:

$n+1$	$(\varphi \to (\theta \to \chi)) \to ((\varphi \to \theta) \to (\varphi \to \chi))$	axioma (ii)
$n+2$	$(\varphi \to \theta) \to (\varphi \to \chi)$	mp
$n+2$	$\varphi \to \chi$	mp

The next example illustrates how the deduction theorem can be used to prove that certain axiomatic deductions exists.

Example Show that $p \to q, q \to r \vdash_a p \to r$.

Proof By the deduction theorem it is sufficient to show that

$$p \to q, q \to r, p \vdash_a r$$

. To this end, consider the next derivation:

1.	p	prem.
2.	$p \to q$	prem.
3.	q	1, 2, mp
4.	$q \to r$	prem.
5.	r	3, 4, mp

Exercise 13 *The proof of the deduction theorem implicitly contains a recipe to construct an axiomatic deduction of $p \to r$ from $p \to q$ and $q \to r$, starting from the deduction given above. Construct such a deduction.*

Exercise 14 *Show that $p \to (q \to r) \vdash_a q \to (p \to r)$.*

Exercise 15 *Show that for languages $\mathcal{L}$ with $\bot$, $\neg$ en $\to$ as the only logical constants the following theorem holds.*

Theorem 3 *$\Delta \vdash_a \varphi$ iff $\Delta \vdash_n \varphi$*

Exercise 16 *Find new axioms such that the theorem from exercise 15 not only holds for languages $\mathcal{L}$ with $\bot, \neg, \to$ as logical constants but also for languages $\mathcal{L}$ with $\bot, \neg, \to, \wedge$, and $\vee$ as logical constants.*

(Hint: try to extrapolate your proof of theorem 3 to the extended languages. The axioms will suggest themselves)