Basic Logic, Homework 1

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Exercise 1

(i) Show that all sentences of the form $((\varphi \to \psi) \to \varphi) \to \varphi$ are tautogies.

(ii) Let p and q be different atomic sentences. Show that $((p \to q) \to q) \to p$ is not a tautology.

(iii) Find a sentence of the form $((\varphi \to \psi) \to \psi) \to \varphi$ that is a tautology.

Exercise 2

Prove or refute each of the following assertions.

(i) If $\Gamma \models \varphi$ or $\Gamma \models \psi$, then $\Gamma \models \varphi \vee \psi$.

(ii) If $\Gamma \models \varphi \vee \psi$, then either $\Gamma \models \varphi$ or $\Gamma \models \psi$.

(iii) $\Gamma, \varphi \models \psi$ iff $\Gamma \models \varphi \to \psi$.

Exercise 3

Prove that every sentence containing only the symbols (,), p, and $\to$ is either a tautology or logically equivalent to the atomic sentence p.

Exercise 4

(i) Show that $p \to q$ is logically equivalent to $\neg p \vee q$.

(ii) Find a sentence, using only the symbols $), (, p, q, \to$, and $\neg$, that is logically equivalent to $p \vee q$.

(iii) Find a sentence, using only the symbols $), (, p, q$, and $\to$, that is logically equivalent to $p \vee q$.

(iv) Prove that there is no sentence in which only the symbols $), (, p, q$, and $\vee$ occur, that is logically equivalent to $p \to q$.

Exercise 5

Let $\mathcal{L}$ be a language of propositional logic with $\wedge$, $\vee$, $\neg$ en $\to$ as logical constants. Call a sentence φ of $\mathcal{L}$ *positive* iff φ is logically equivalent to a sentence in which $\wedge$ and $\vee$ are the only logical constants.

(a) Show that $\neg(p \to \neg q)$ is a positive sentence.

(b) Prove that no positive sentence is a tautology.

(c) Let I and J be interpretations. Define:
$I \preceq J$ iff for all proposition symbols p such that $I(p) = 1$, it holds that $J(p) = 1$.
Show that for all positive sentences φ the following holds: If $I \preceq J$ and $V_I(\varphi) = 1$, then $V_J(\varphi) = 1$.

(d) Show that $\neg((p \to q) \to \neg r)$ is not positive.